# *All Bent Out of Shape: The Dynamics of Thin Viscous Sheets*

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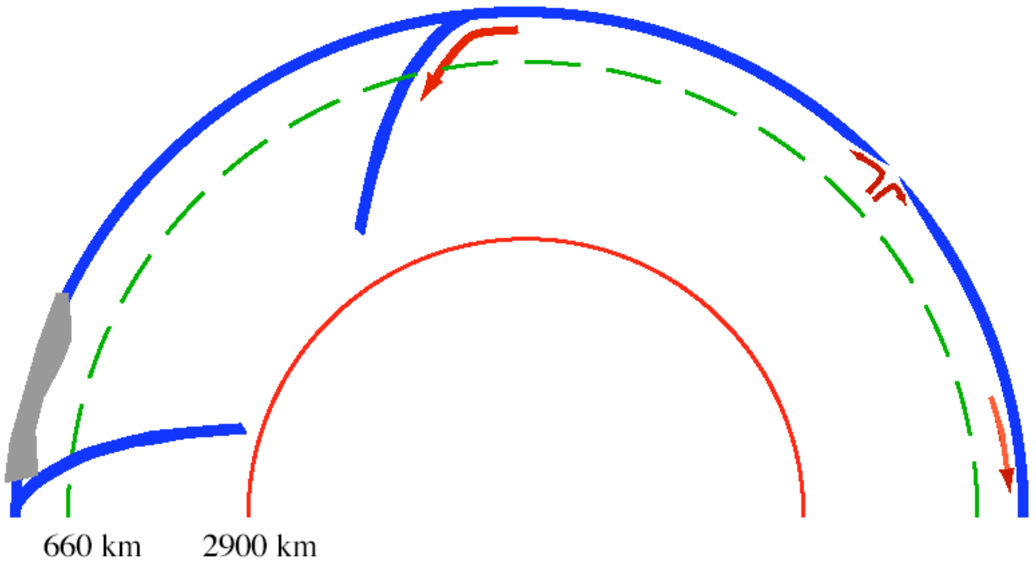
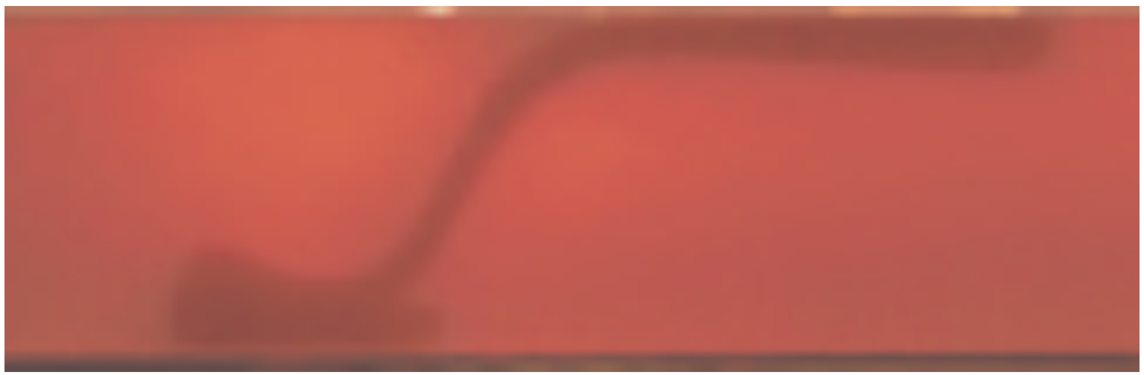
Photo courtesy of Mars, Inc.

# *Modeling Earth's lithosphere as a thin sheet*

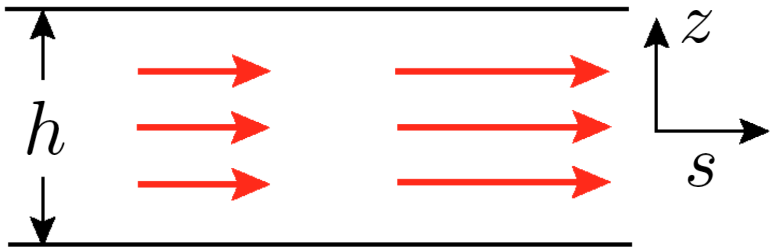



Analog experiments on subduction (Roma-III):



## *Thin viscous sheets: two modes of deformation*

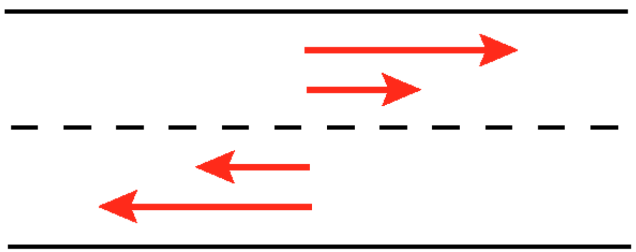
### 1. Stretching :




### Stress resultant :

$$N \equiv \int_{-h/2}^{h/2} \sigma_{ss}\,\mathrm{d}z \quad = \underbrace{4\eta}_{\text{Trouton viscosity}}\, h\, \underbrace{\Delta}_{\text{rate of stretching}}$$

### 2. Bending :



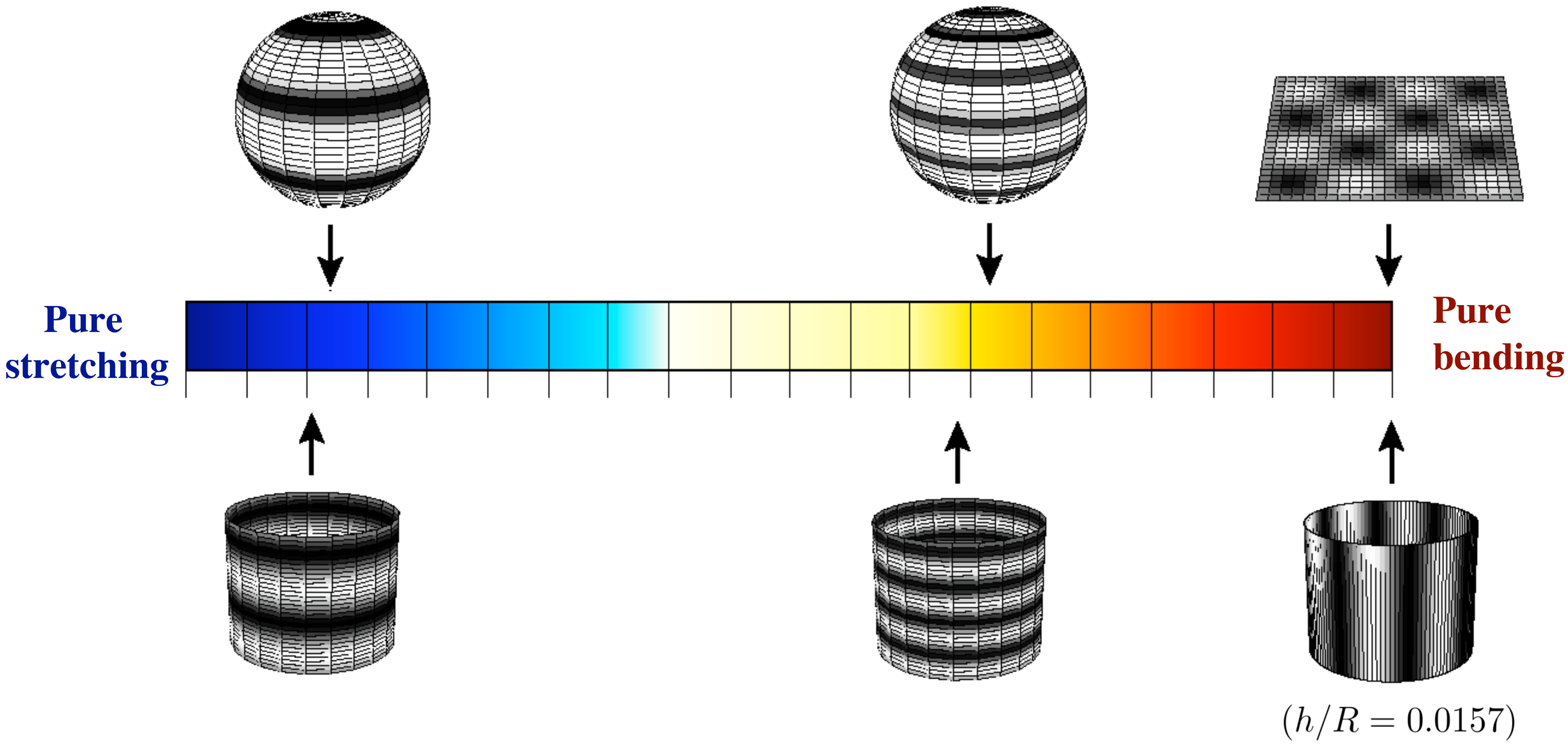
### Bending moment :

$$M \equiv \int_{-h/2}^{h/2} z\sigma_{ss}\,\mathrm{d}z \quad = \underbrace{\frac{1}{3}\eta\, h^3}_{\text{viscous flexural rigidity}}\, \underbrace{\dot{K}}_{\text{rate of change of curvature}}$$

# *Loaded viscous sheets: bending vs. stretching*

Partitioning depends on: (1) sheet shape (curvature)
(2) loading distribution
(3) edge conditions

**Example:** periodic normal loading (no edges)




***Rule of thumb:*** stretching $\approx$ bending when $\lambda \approx 4(hR)^{1/2}$ ($\approx$ 3200 km on Earth)

## *Intermediate length scales in viscous sheet dynamics*

Competition of bending and stretching

↓

length scales that are intermediate between the

sheet's thickness and its lateral dimension

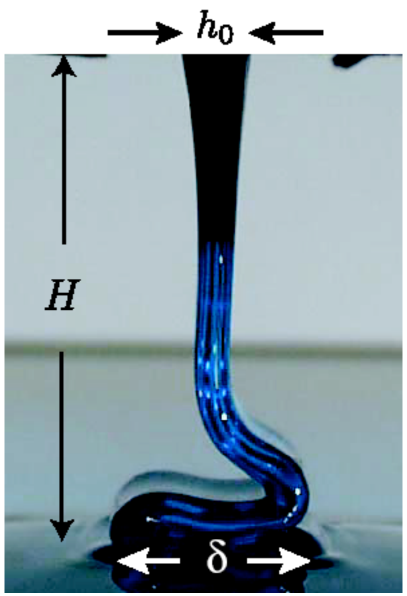
*Examples :*

- **normally loaded spherical or cylindrical sheets :**

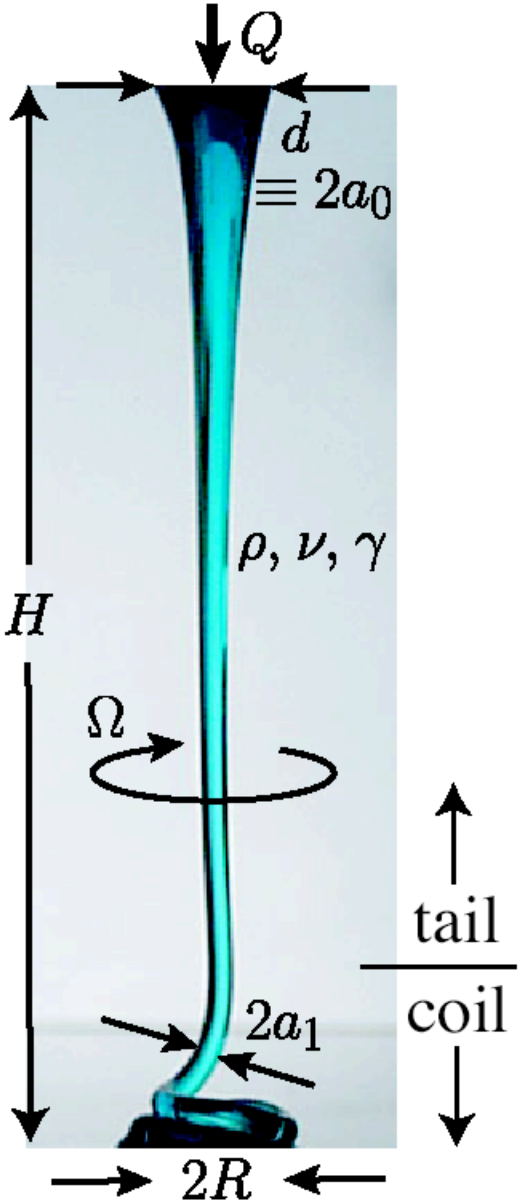
  stretching/bending transition occurs at load wavelength

  $$\lambda \sim (hR)^{1/2}$$

- **periodic buckling instabilities :**



# *An analogous system: coiling of a viscous « rope »*

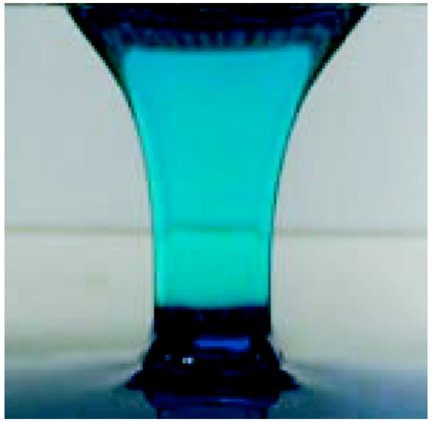



**Why coiling is simpler than folding:**

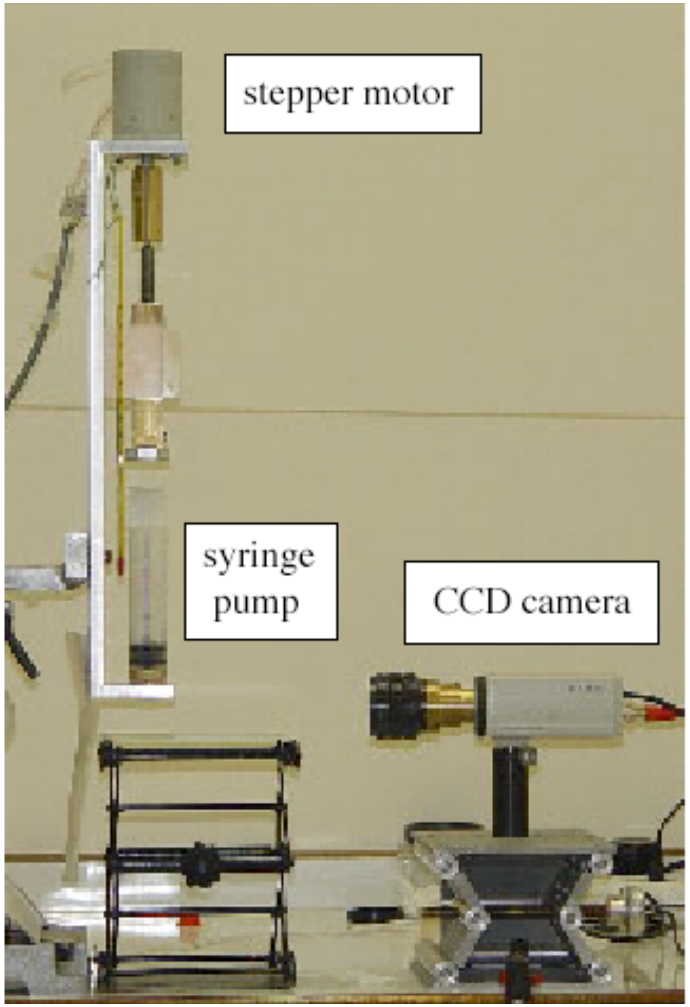
- folding is inherently unsteady
- folding sheets contract in the transverse direction :

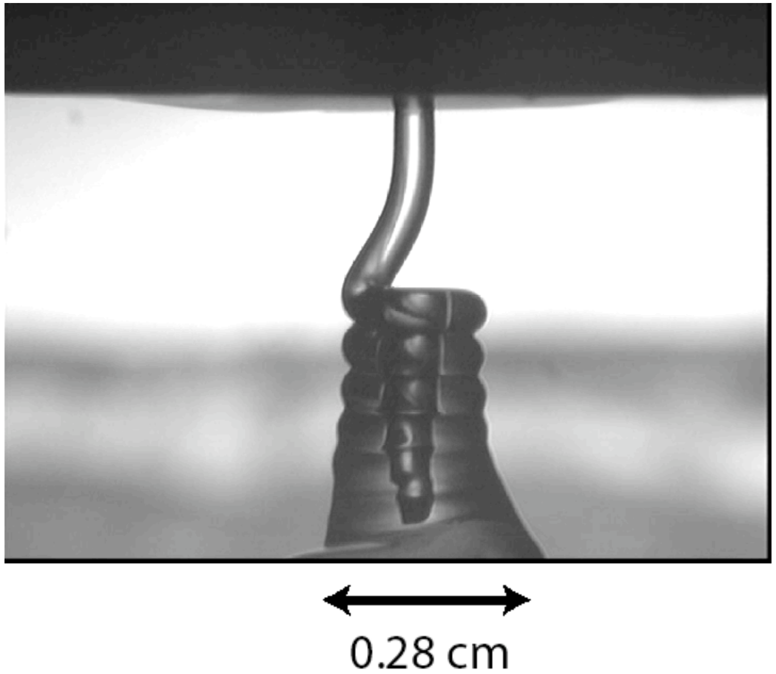
edge-on view:

side view:

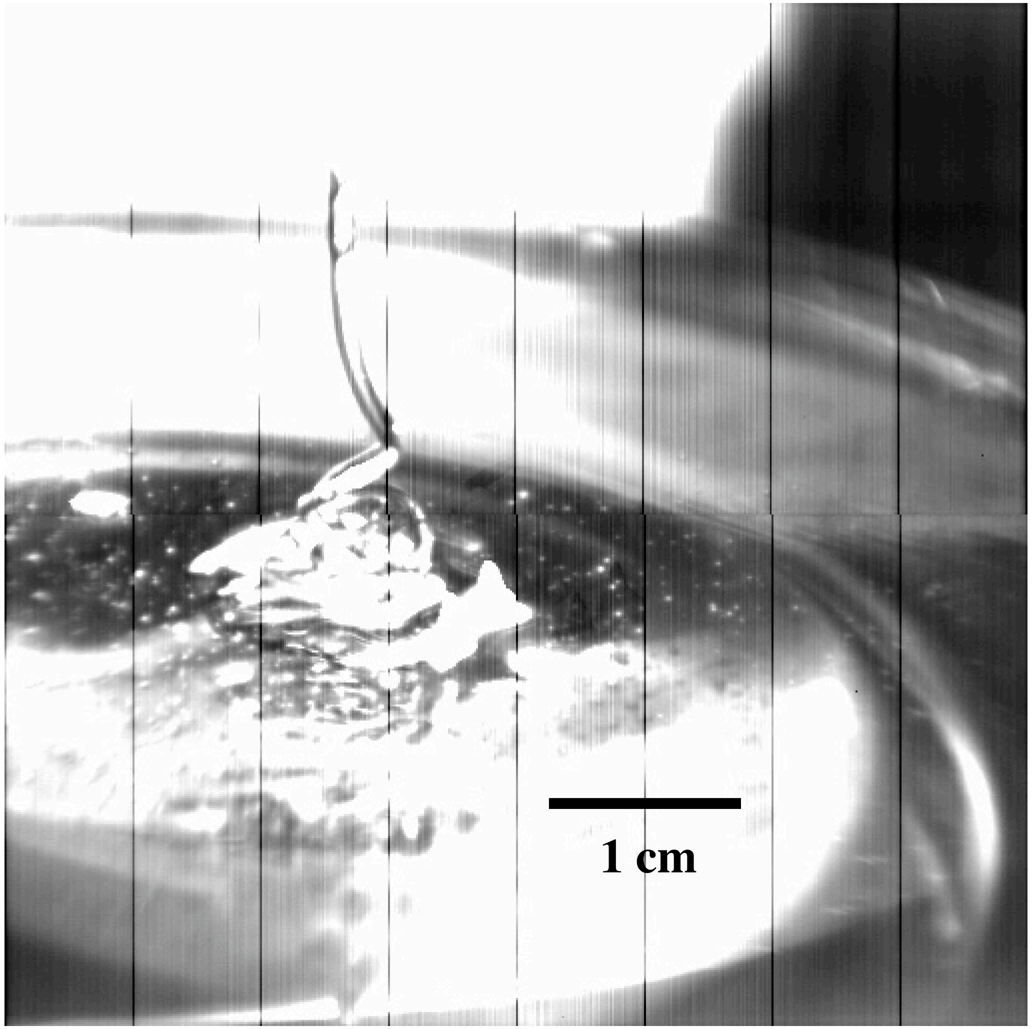


# *Experimental setup*

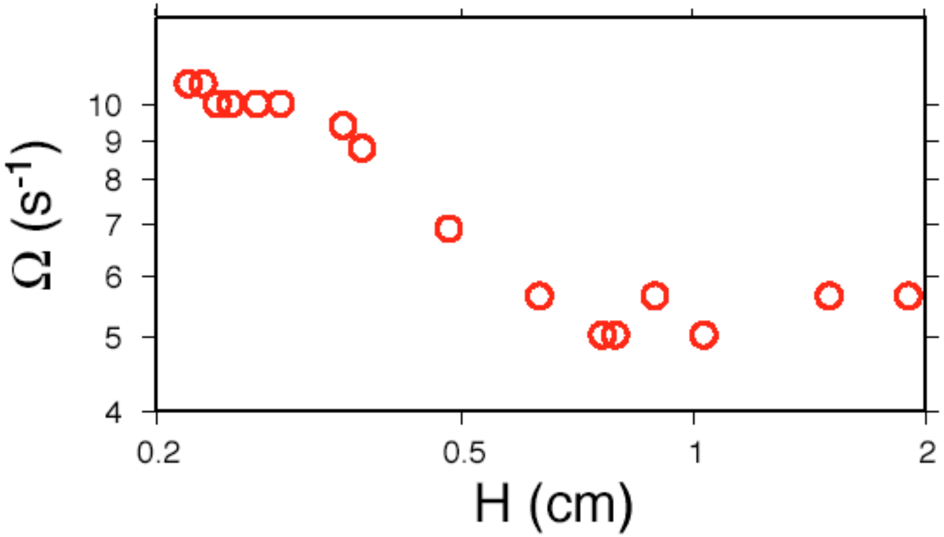


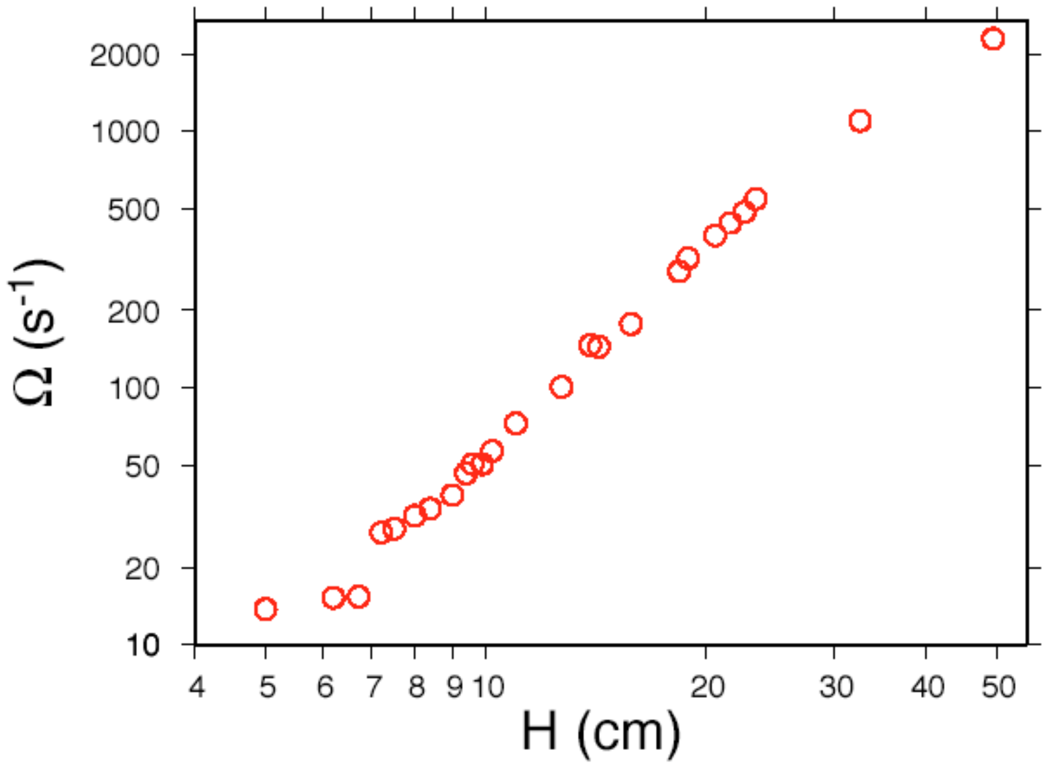



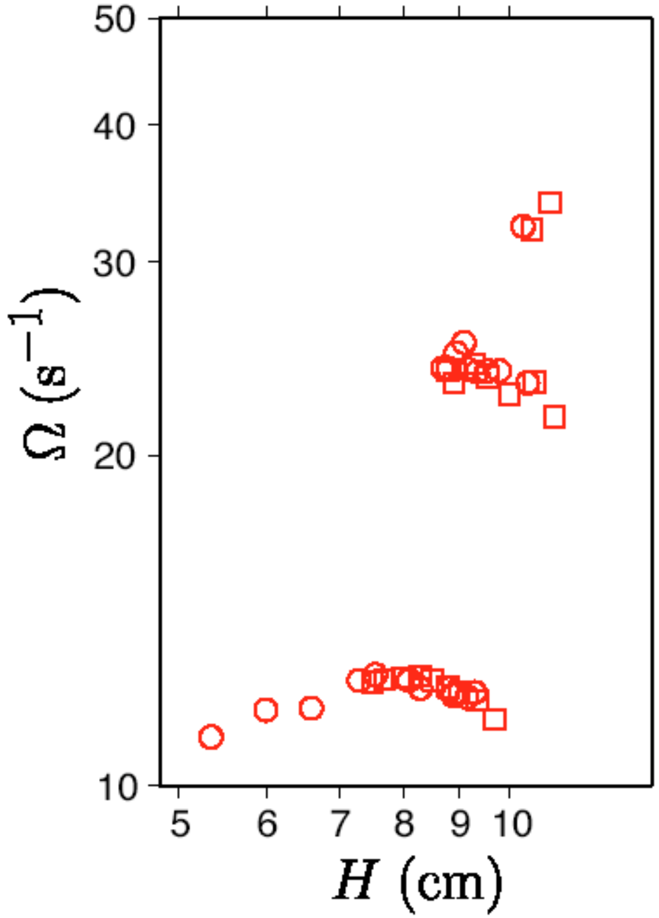
# *Multistable coiling*



***Coiling frequency vs. height: Experimental observations***



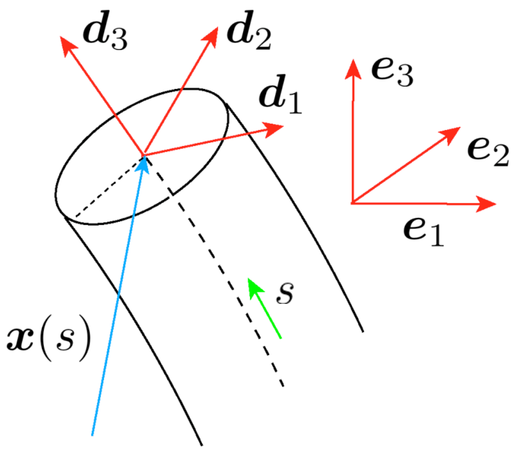



# *Steady coiling: mathematical formulation*

(Ribe 2004, Proc. R. Soc. Lond. A)

- *17 variables:*
  - $\boldsymbol{x}$ coordinates of axis
  - $q_0,\ q_1,\ q_2,\ q_3$ Euler parameters
  - $\kappa_1,\ \kappa_2$ curvatures of axis
  - $U$ axial velocity
  - $\omega_3$ spin about axis
  - $\boldsymbol{N} = \int_S \boldsymbol{\sigma} \cdot \boldsymbol{d}_3 \, dS$ stress resultant vector
  - $\boldsymbol{M} = \int_S \boldsymbol{r} \times (\boldsymbol{\sigma} \cdot \boldsymbol{d}_3) \, dS$ bending moment vector

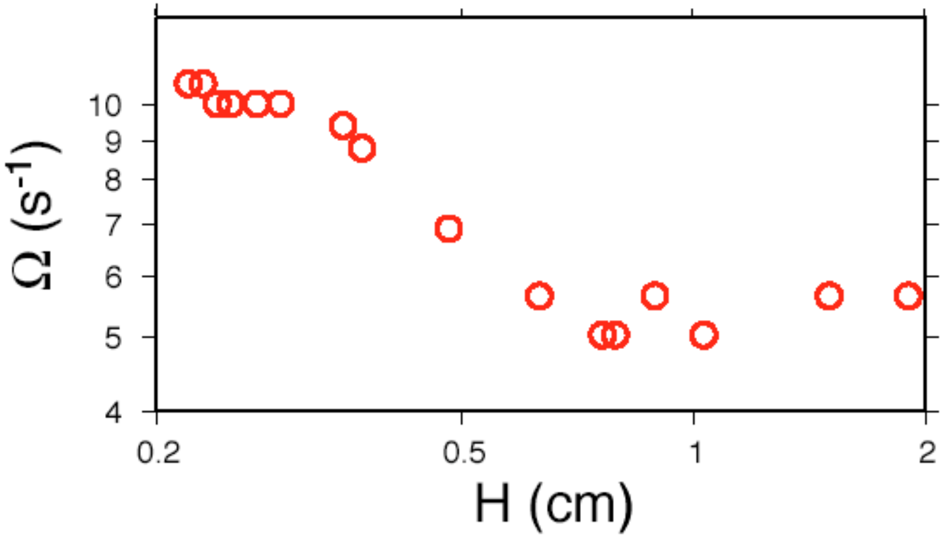


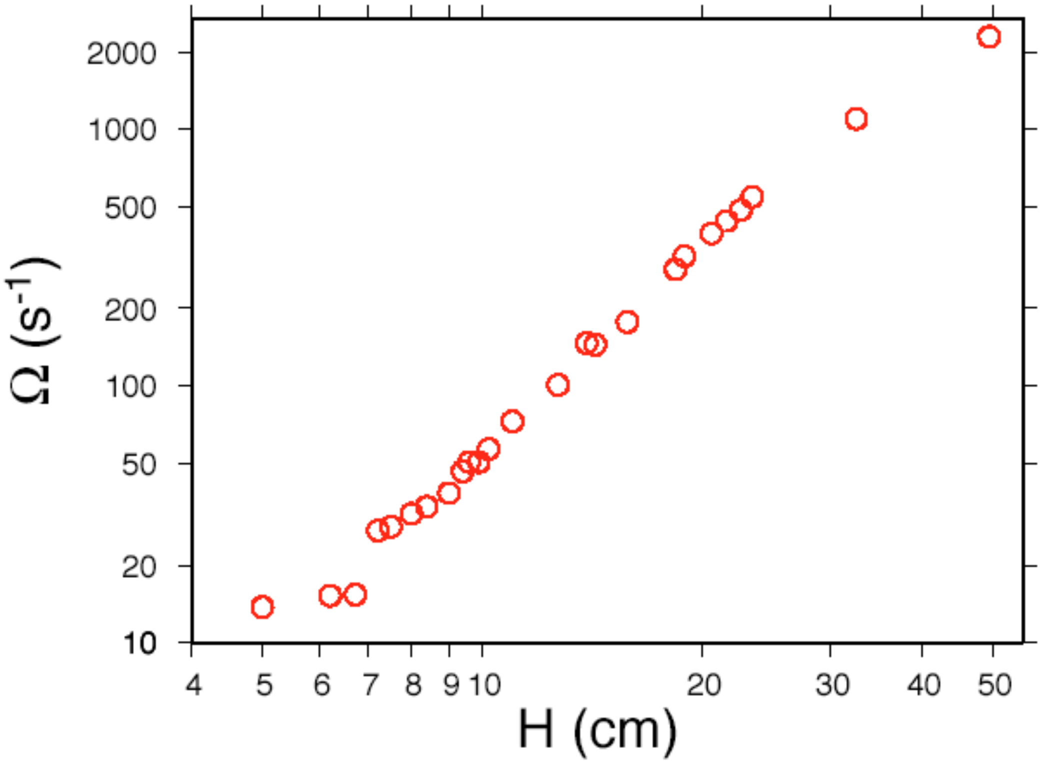

- *17 governing equations* (3 force balance; 3 torque balance; 7 geometric; 4 constitutive relations)
- *2 unknown parameters:*
  - $\Omega$ coiling frequency
  - $\ell$ filament length
- *19 boundary conditions*

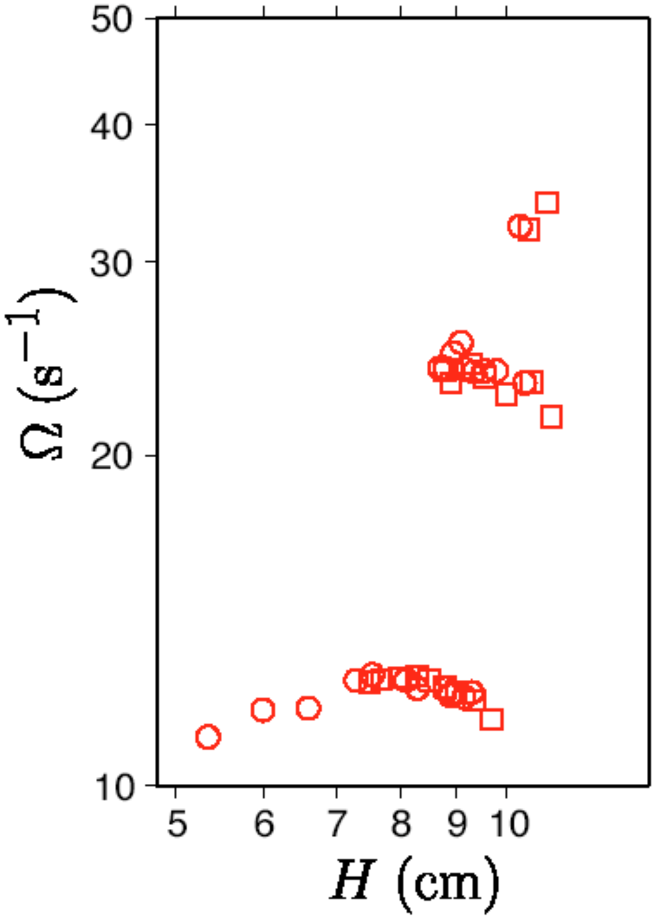
→ 17th-order nonlinear boundary value problem

- *Numerical method:* continuation (AUTO 97; Doedel *et al*. 2002)

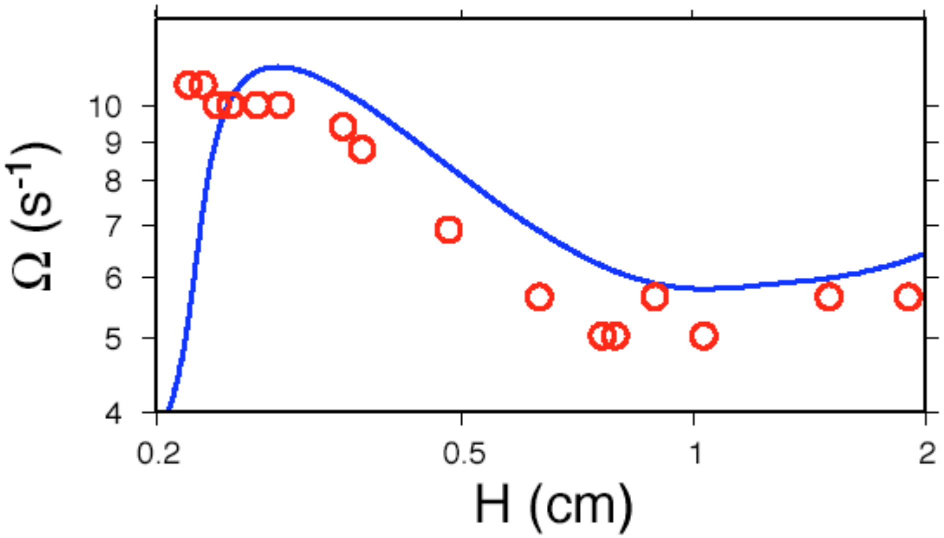
# *Coiling frequency vs. height: Experimental observations*

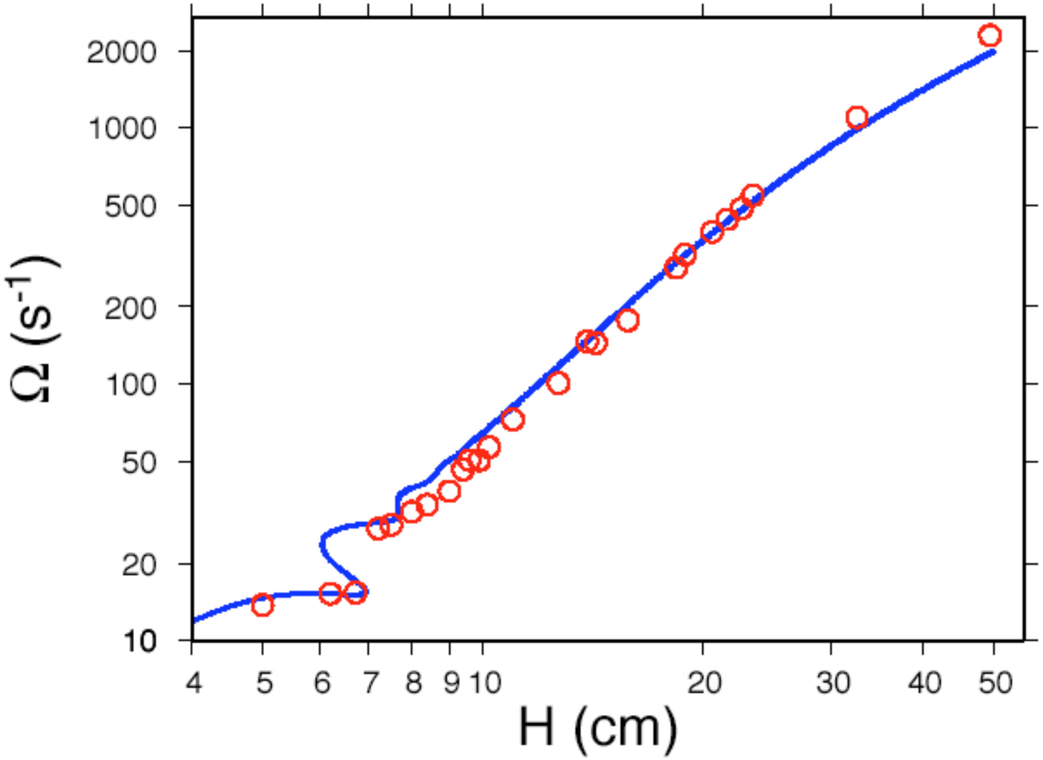


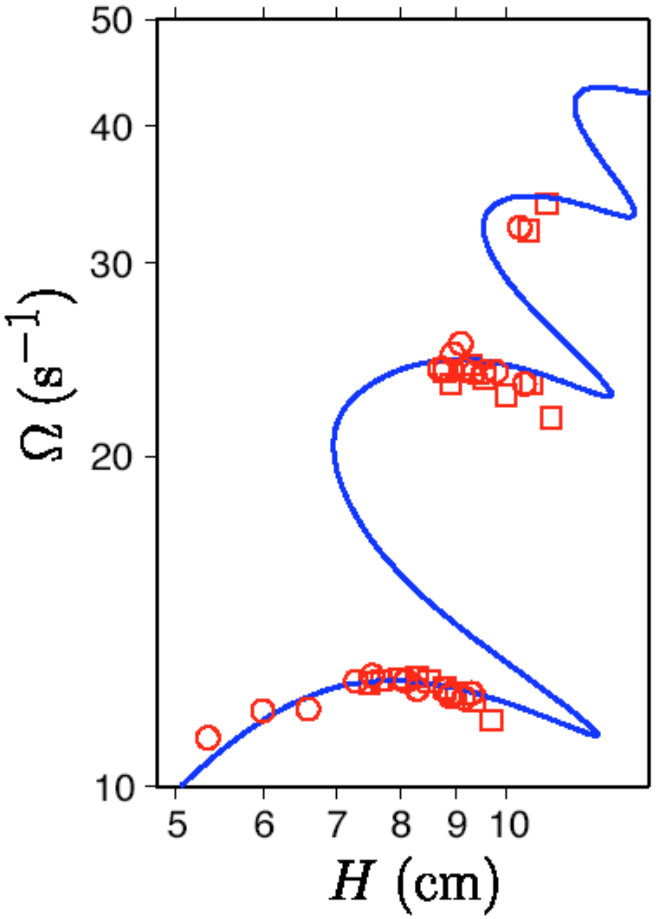



## *Comparison with numerical solutions*

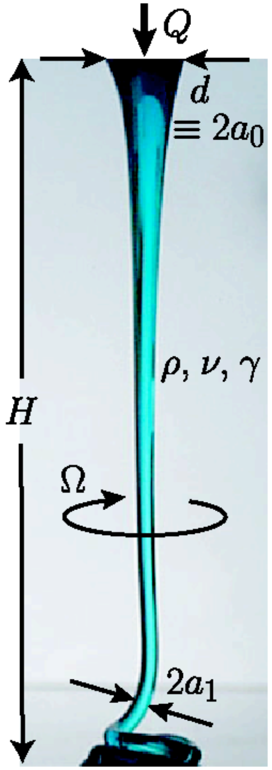


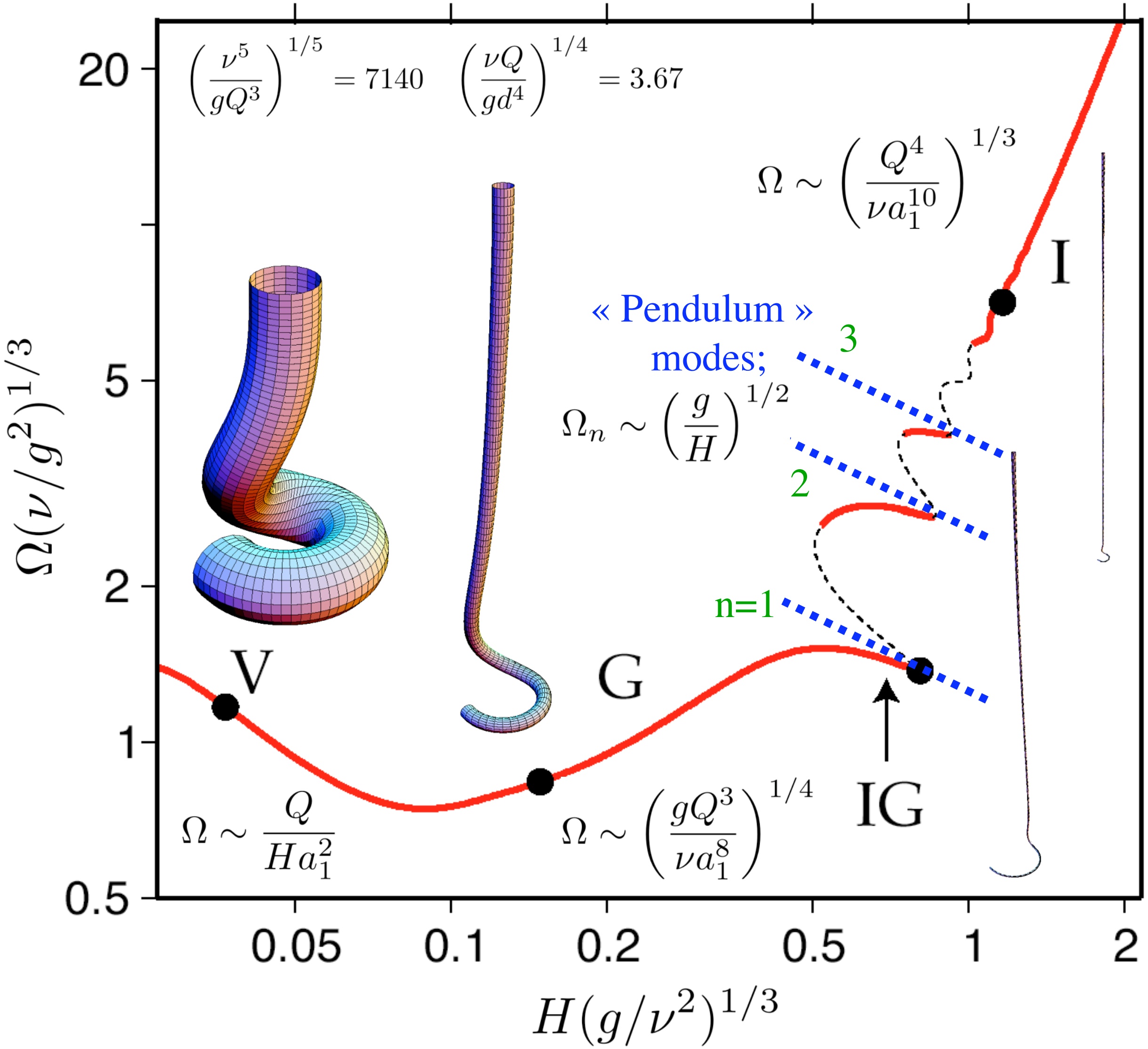



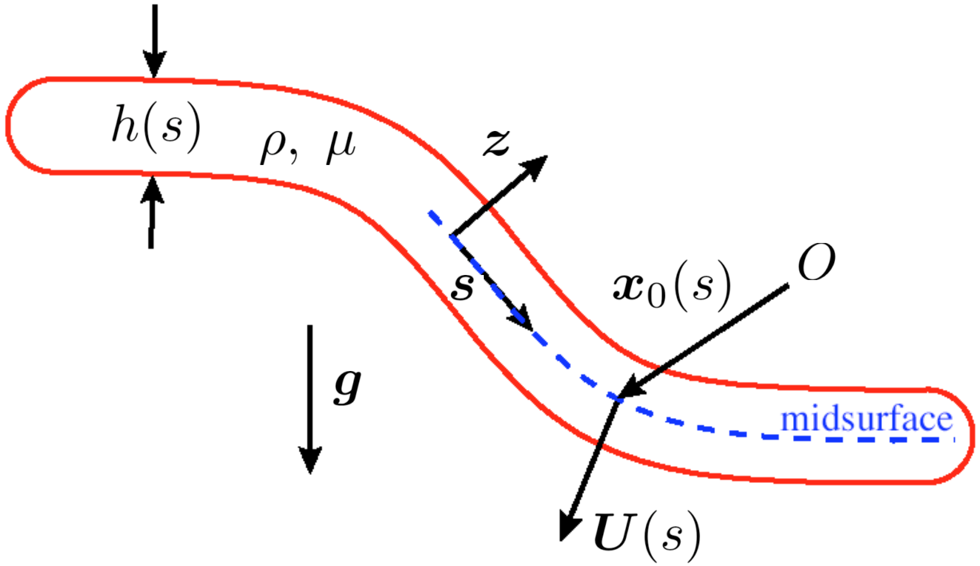
## *Coiling frequency vs. height: Four regimes*

(Mahadevan et al., Nature 2000;
Ribe, Proc. R. Soc. Lond. 2004;
Maleki et al., Phys. Rev. Lett. 2004;
Ribe et al., J. Fluid Mech. 2006;
Ribe et al., Phys. Fluids 2006)



# *Slow deformation of thin viscous sheets: governing equations (2D)*




**Force balance:**

$$\underbrace{\boldsymbol{N}'}_{\text{viscous forces}} + \underbrace{h\rho\boldsymbol{g}}_{\text{gravity}} = 0 \qquad \left(' = \frac{\mathrm{d}}{\mathrm{d}s}\right)$$

**Viscous stress resultant:**

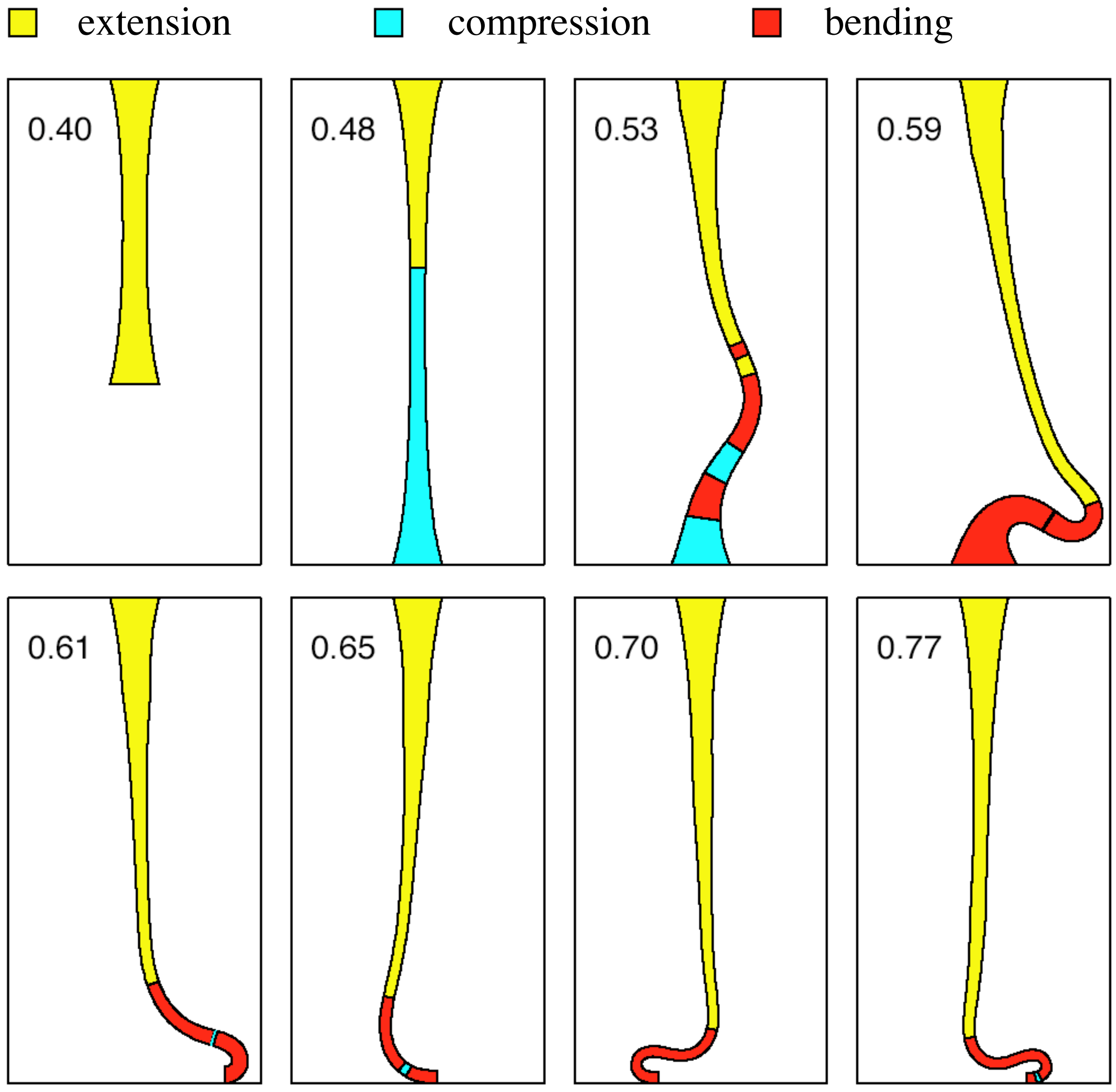
$$\boldsymbol{N} \equiv \int_{-h/2}^{h/2} \boldsymbol{\sigma}\cdot\boldsymbol{s}\,\mathrm{d}z = \underbrace{4\mu h(\boldsymbol{U}'\cdot\boldsymbol{s})\boldsymbol{s}}_{\text{stretching}} + \frac{\mu}{3}\underbrace{[h^3(\boldsymbol{U}'\cdot\boldsymbol{z})']'\boldsymbol{z}}_{\text{bending}}$$

**Evolution equations for the sheet's shape:**

$$\frac{D\boldsymbol{x}_0}{Dt} = \boldsymbol{U}, \qquad \frac{Dh}{Dt} = -h\boldsymbol{U}'\cdot\boldsymbol{s}$$

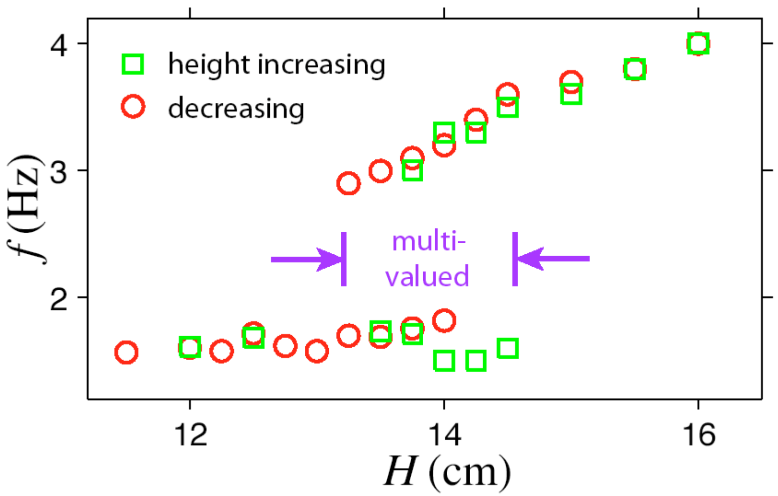
# *Development of a buckling instability*

(numerical simulation; Ribe, J. Fluid Mech. 2002)

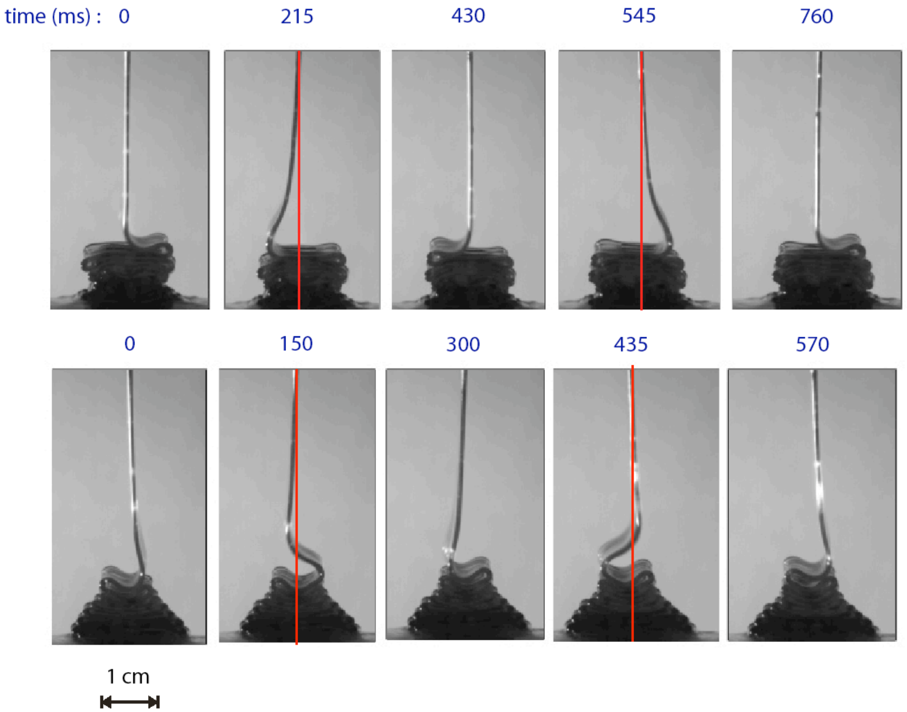


# *Multivalued folding*

**Frequency vs. height :**



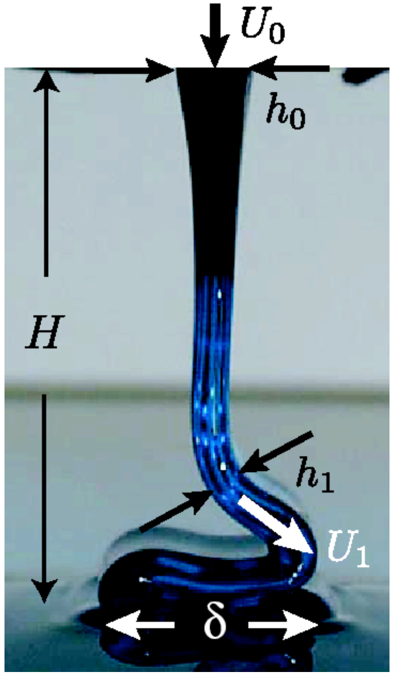

**First two « pendulum » modes :**

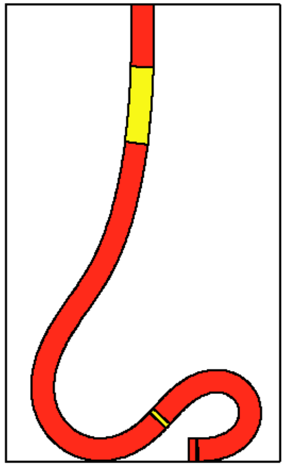
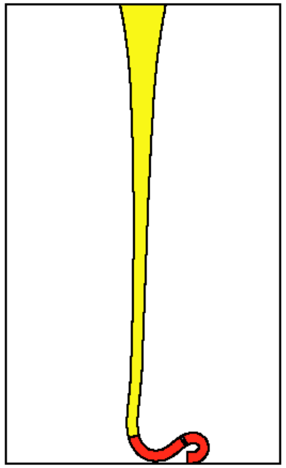


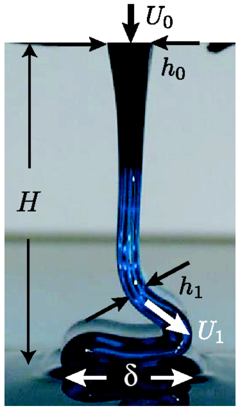

**Mode 1**

**Mode 2**

# *Two modes of slow (inertia-free) folding*



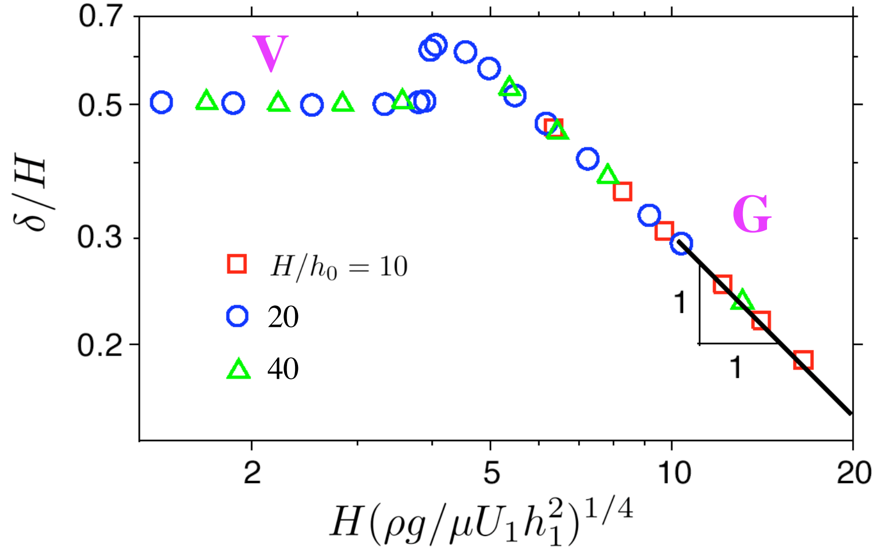

| *Mode* | *Amplitude δ* | *Numerical simulation* |
|---|---|---|
| Viscous (V) | $\sim H$ |  |
| Gravitational (G) | $\sim \left(\frac{\mu h_1^2 U_1}{\rho g}\right)^{1/4}$ (Skorobogatiy & Mahadevan 2000) |  |



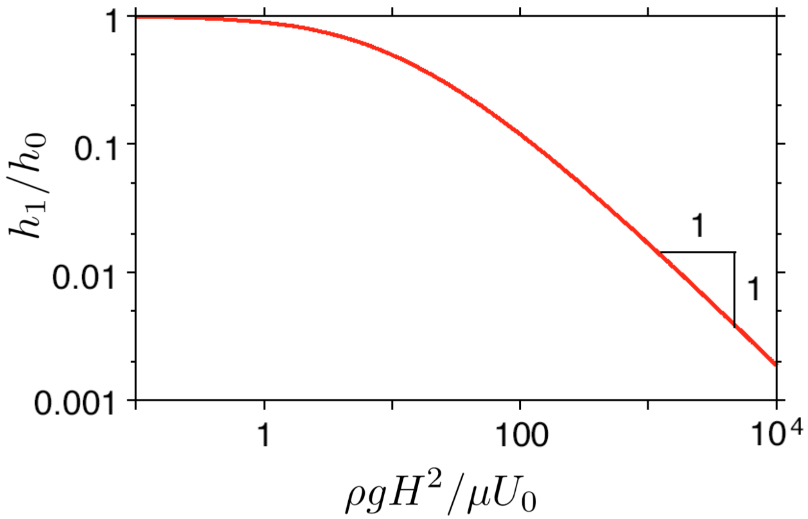

# *Folding amplitude: Universal scaling law*

(Ribe, Phys. Rev. E 2003)

**1.** fold amplitude: $\dfrac{\delta}{H} = F_1\left( H\left[\dfrac{\rho g}{\mu U_1 h_1^2}\right]^{1/4}\right)$



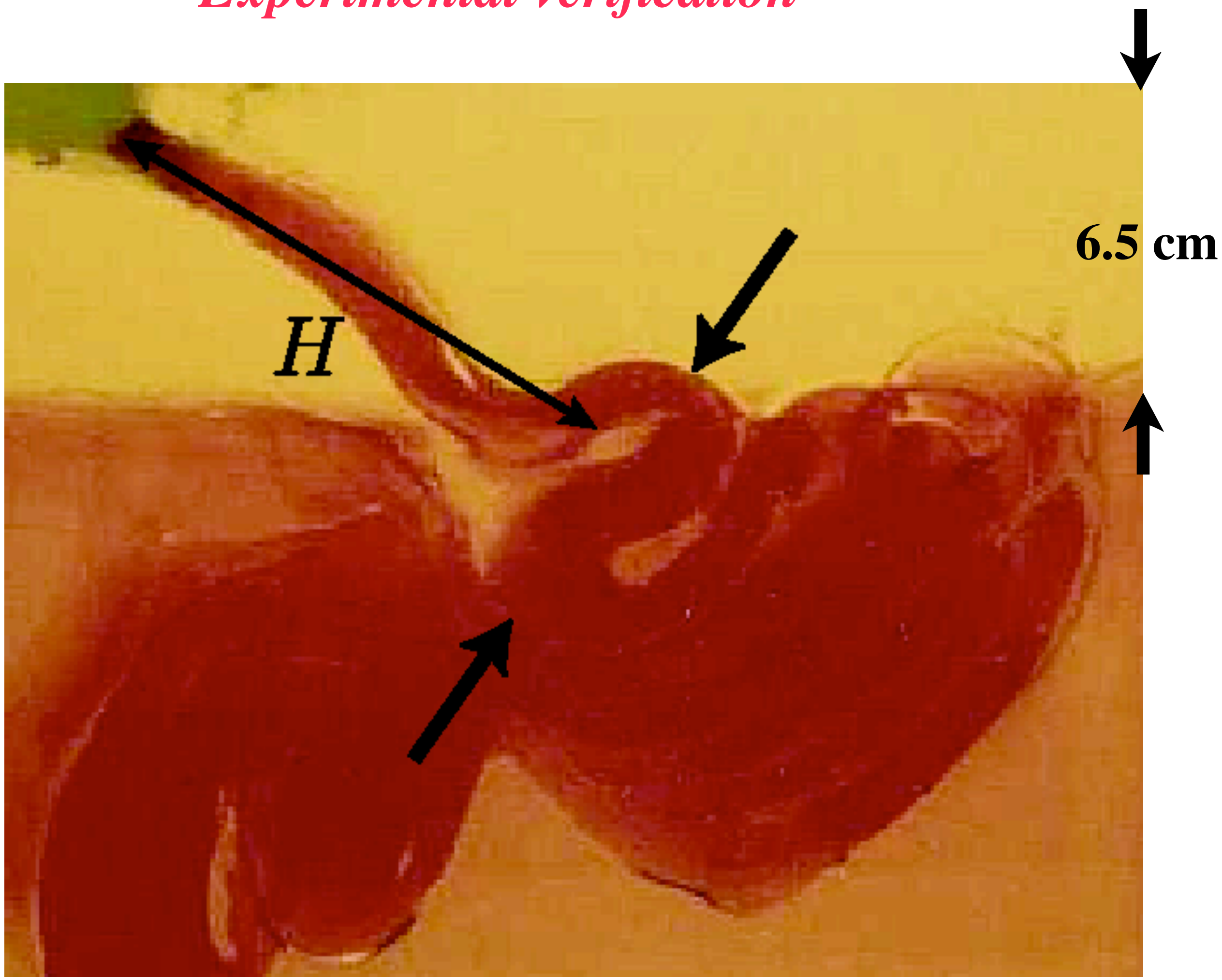

**2.** stretching in the tail: $\dfrac{h_1}{h_0} = F_2\left(\dfrac{\rho g H^2}{\mu U_0}\right)$




**Composite scaling law:**

$$\delta = H F_1\left( H\left[\frac{\rho g}{\mu U_0 h_0^2 F_2(\rho g H^2/\mu U_0)}\right]^{1/4}\right)$$

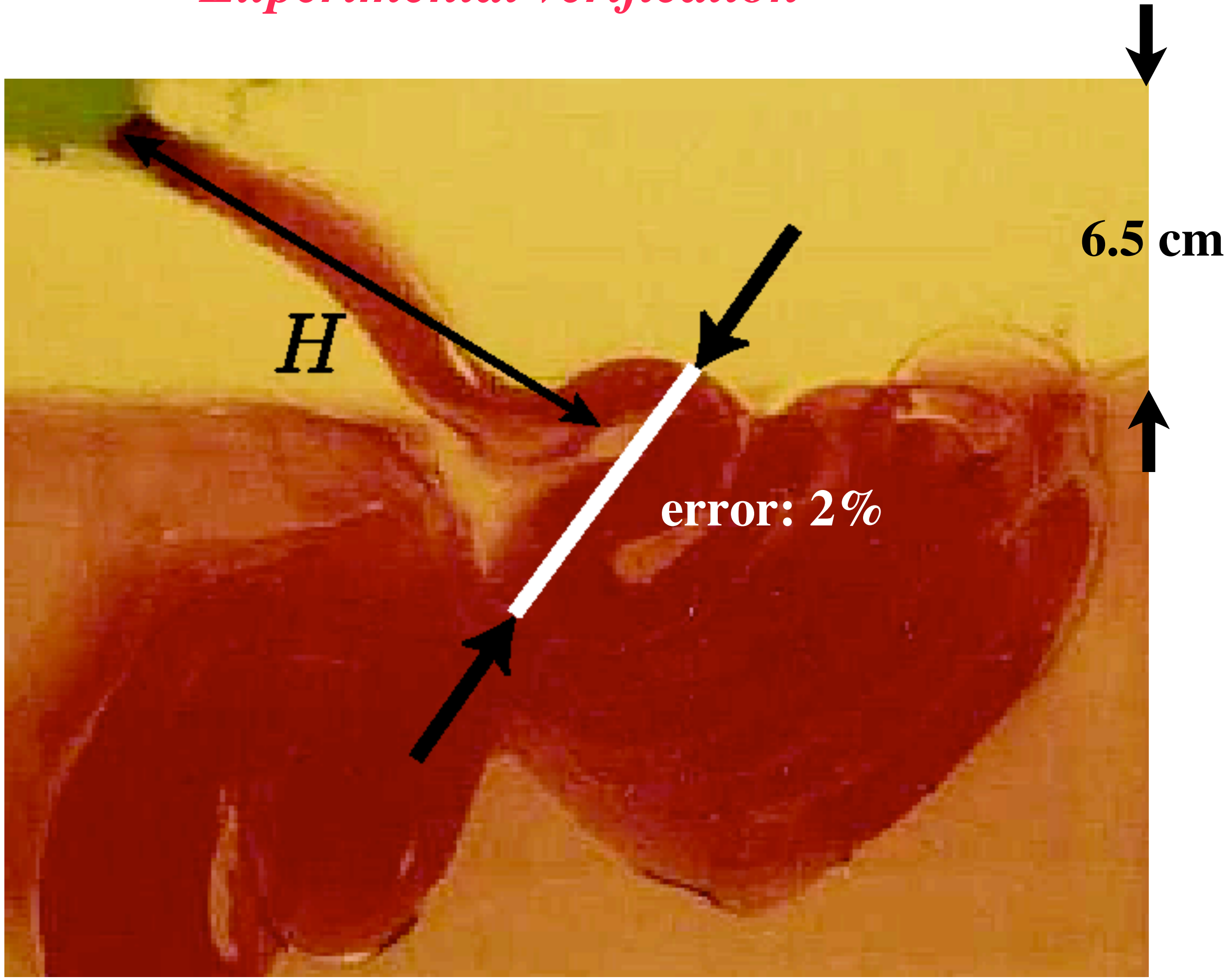
*Scaling law for folding amplitude:*
*Experimental verification*




Guillou-Frottier
et al. 1995

| $\Delta\rho$ (kg/m$^3$) | $\mu$ (Pa s) | $U_0$ (cm/s) | $h_0$ (cm) | dip $\theta$ (deg.) | $\delta$ (cm) |
|---|---|---|---|---|---|
| 58 | 7 X $10^5$ | 0.05 | 1.0 | 35 | **5.7** |

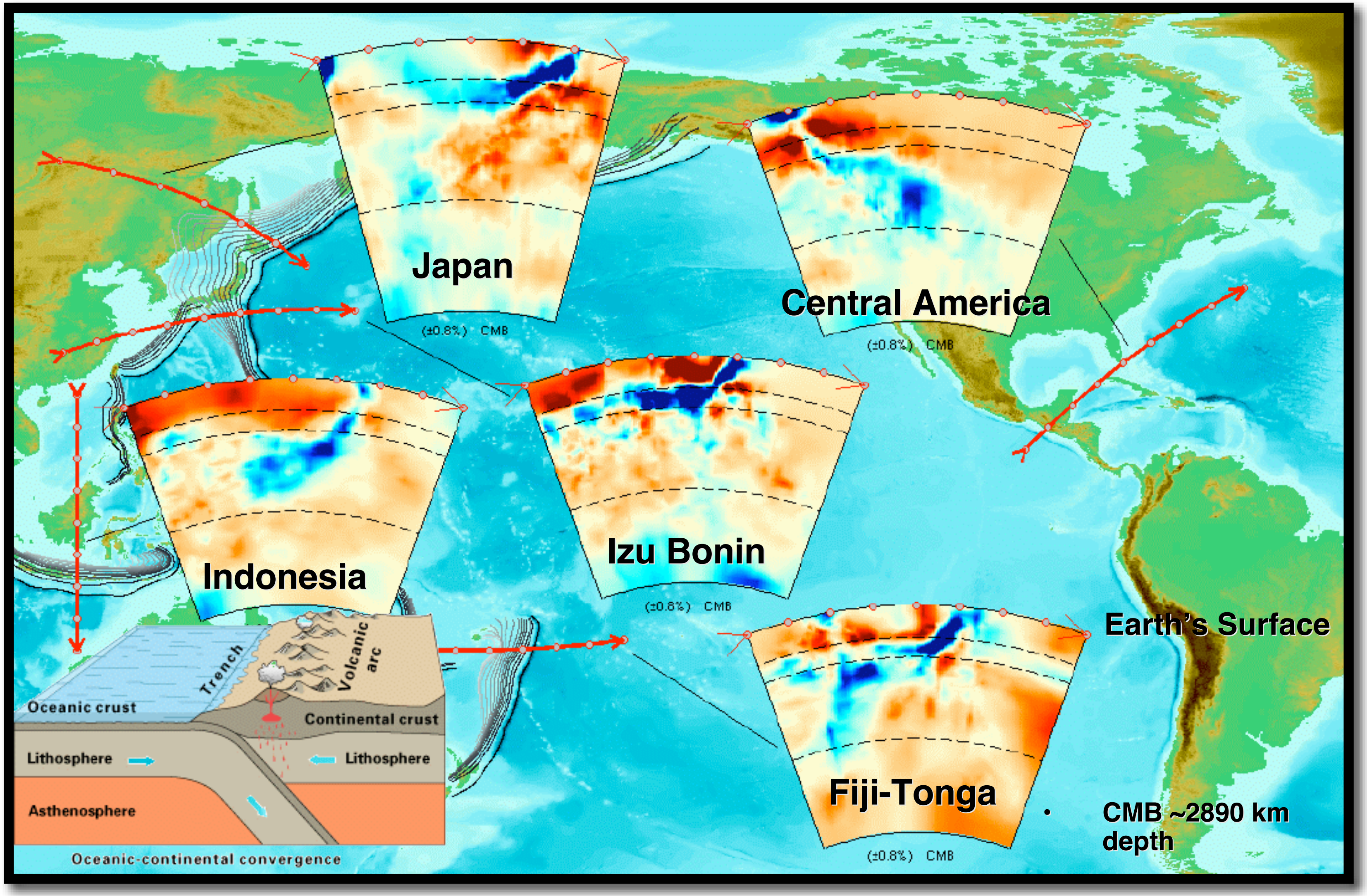
# *Scaling law for folding amplitude: Experimental verification*




Guillou-Frottier et al. 1995

| $\Delta\rho$ (kg/m$^3$) | $\mu$ (Pa s) | $U_0$ (cm/s) | $h_0$ (cm) | dip $\theta$ (deg.) | $\delta$ (cm) |
|---|---|---|---|---|---|
| 58 | 7 X $10^5$ | 0.05 | 1.0 | 35 | **5.7** |

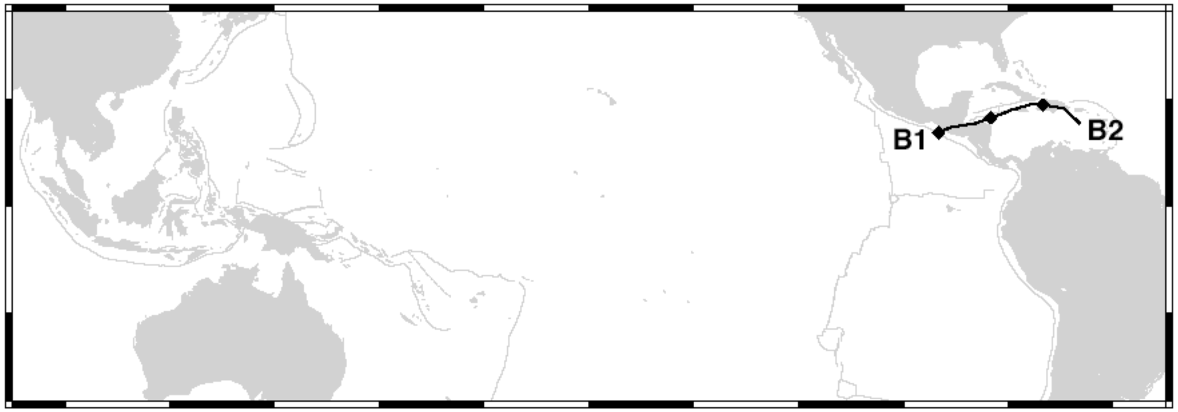
# *Tomographic images of subducted slabs*

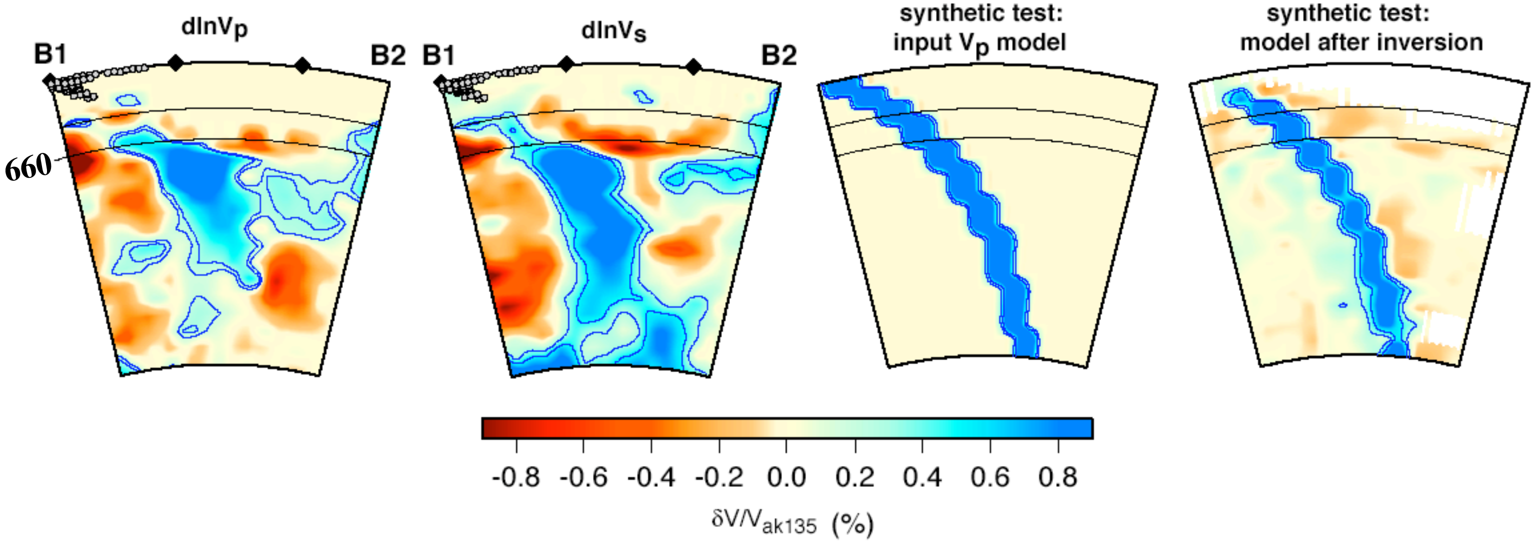



Albarède and van der Hilst, Phil. Trans. R. Soc. Lond. A, 2002

# *Case study : Central America*

**Regional seismic tomography:**
(Ren et al. 2006)



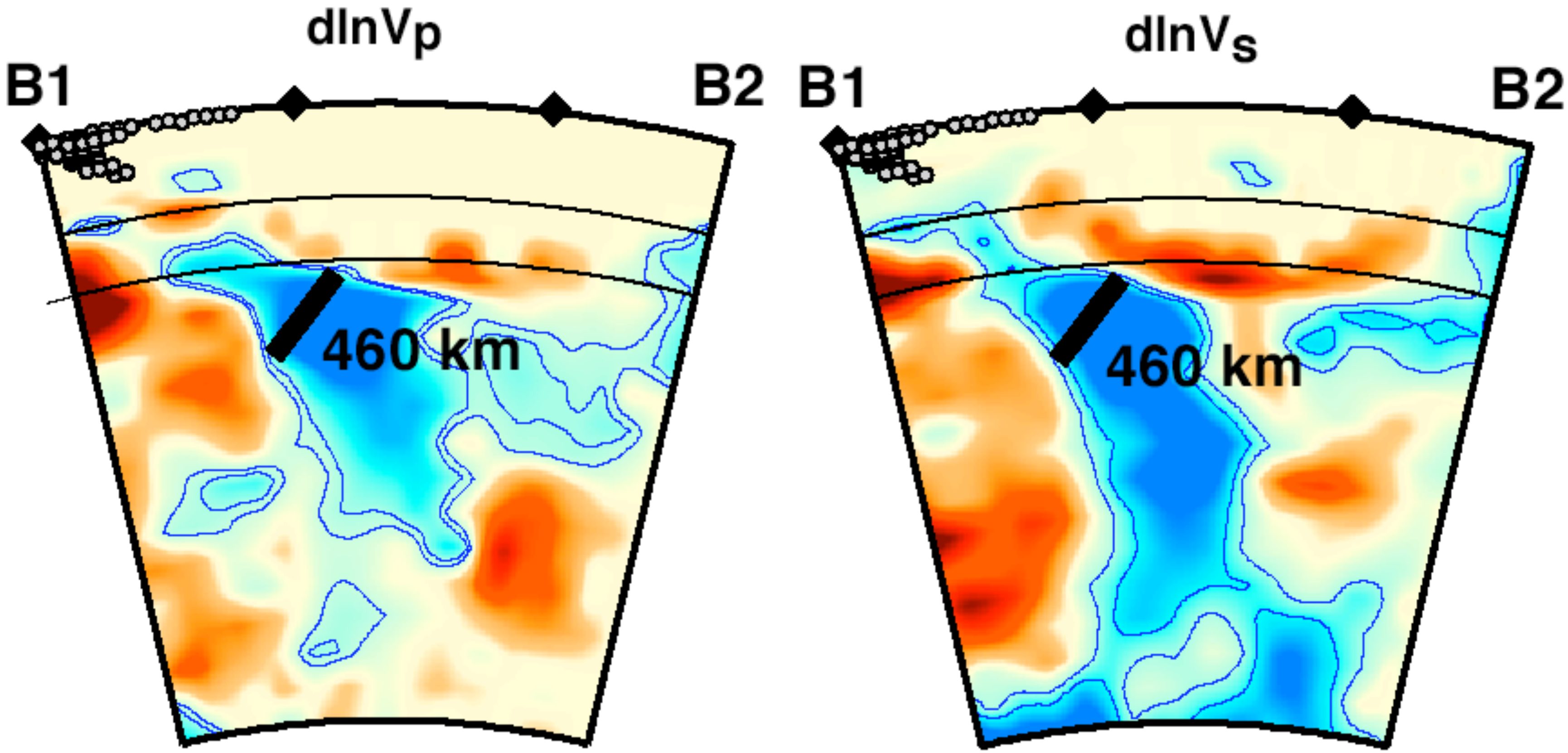




**Predicted folding amplitude:**

| $\Delta\rho$ (kg/m$^3$) | $\mu$ (Pa s) | $U_0$ (cm/yr) | $h_0$ (km) | dip $\theta$ (deg.) | $\delta$ (km) |
|---|---|---|---|---|---|
| 65 | $10^{23}$ | 6.3 | 45 | 65 | **460** |

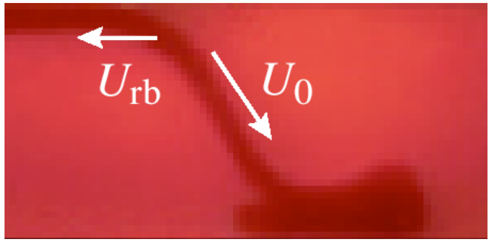
***Central America :***
***predicted fold amplitude***
***vs.***
***observed width of tomographic anomalies***

(Ribe et al., EPSL 2007)



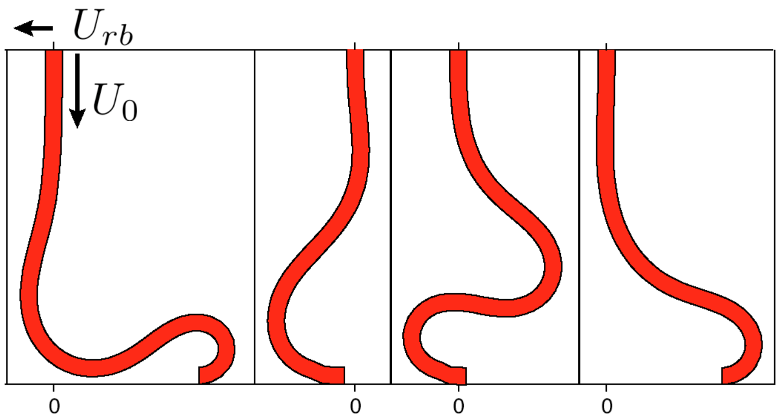
# *Effect of trench rollback on buckling*

**Parameters :**

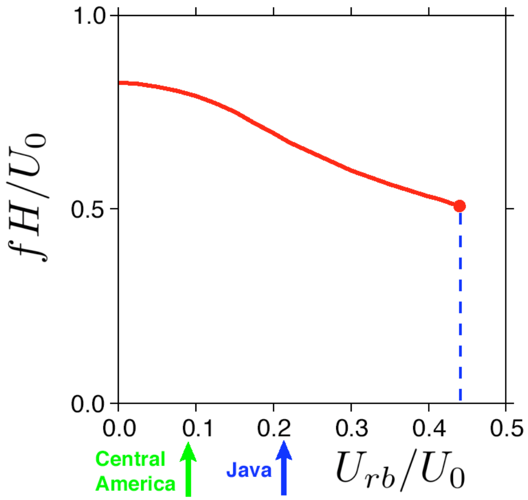



**Numerical model of buckling with rollback**

$(U_{rb} = 0.3U_0)$




**Buckling frequency vs. rollback speed**



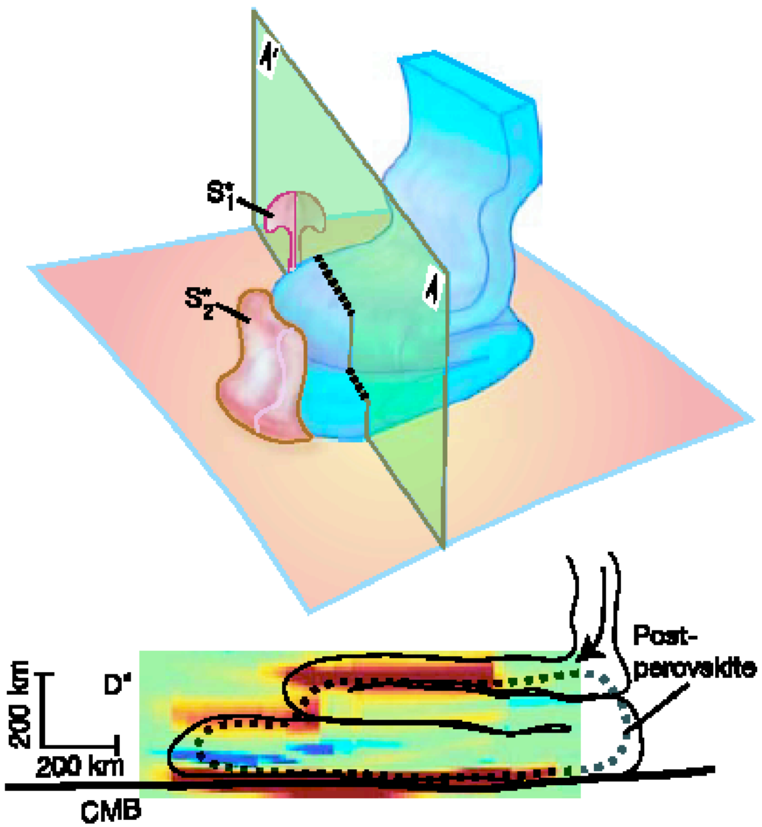

Buckling ceases when

$U_{rb} > 0.44U_0$

# *Folding of subducted lithosphere at the CMB?*

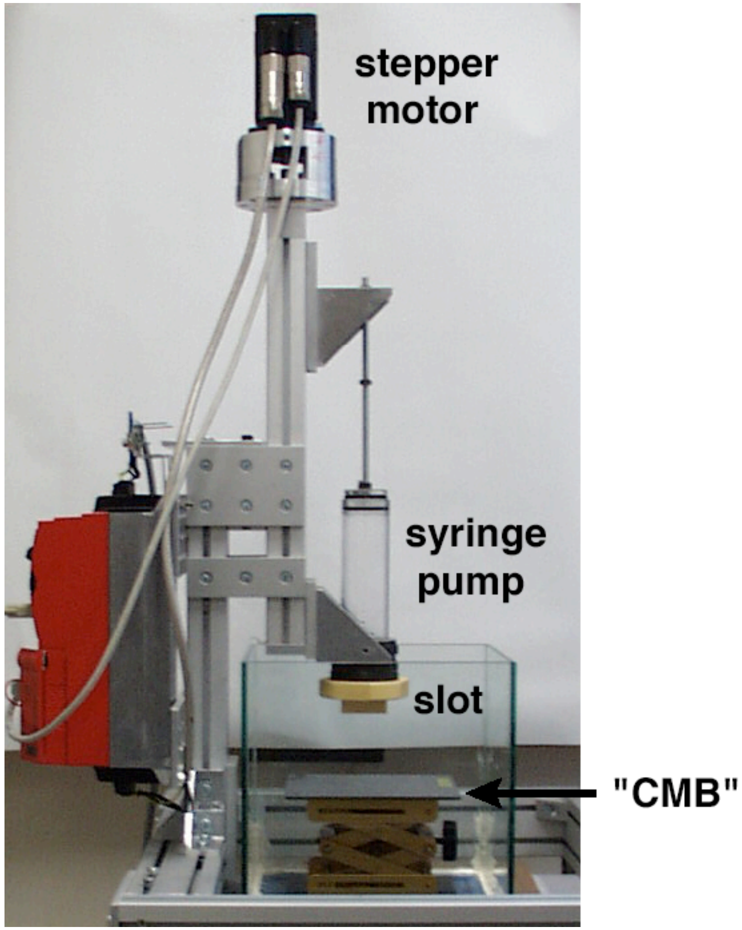
**Seismic observations of folding (?) beneath Central America**

(Hutko et al. 2006)

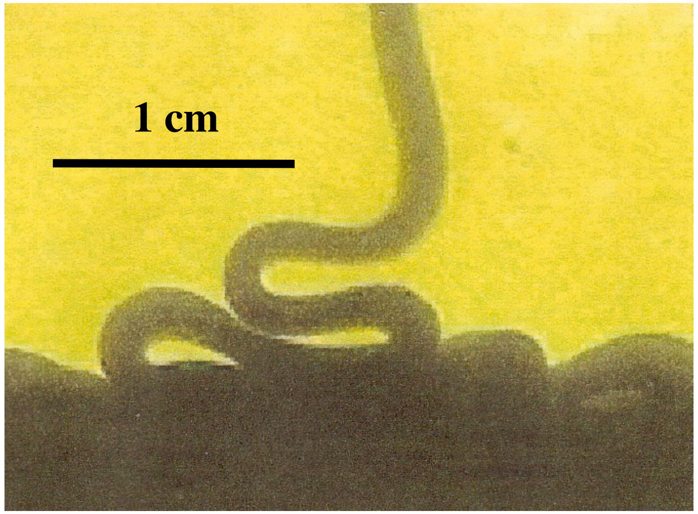



**Experimental setup**

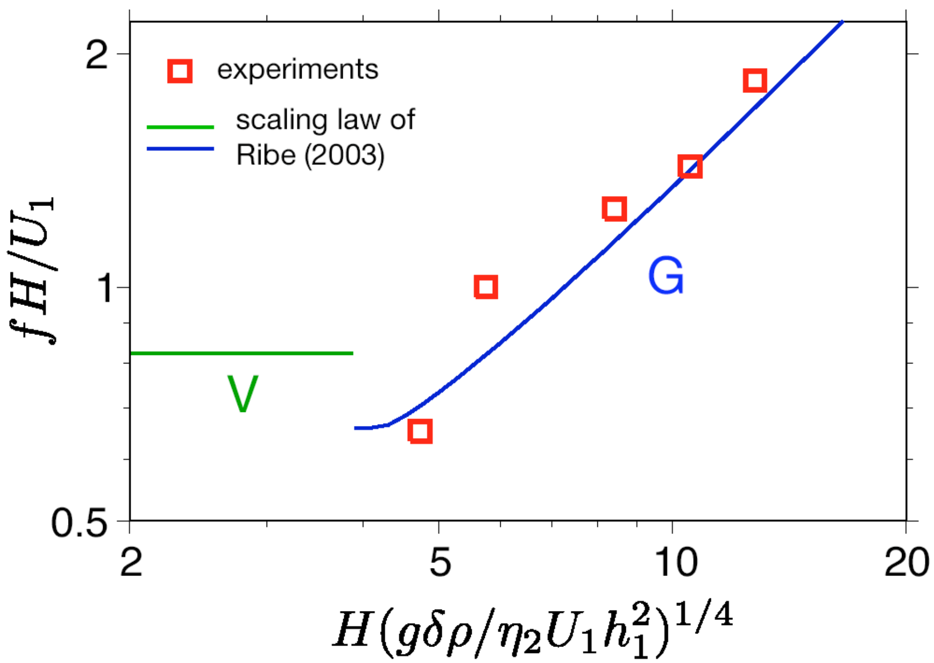
**(IPGP)**



## *Limit 1: High viscosity contrast* $(\eta_2/\eta_1 = 485)$



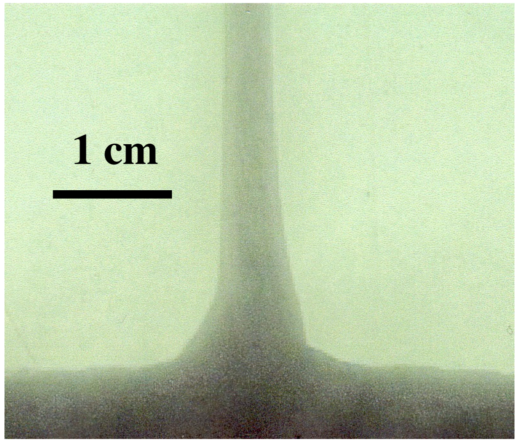

Rescaled folding frequencies:




➡ **Folding is unaffected by the ambient fluid**

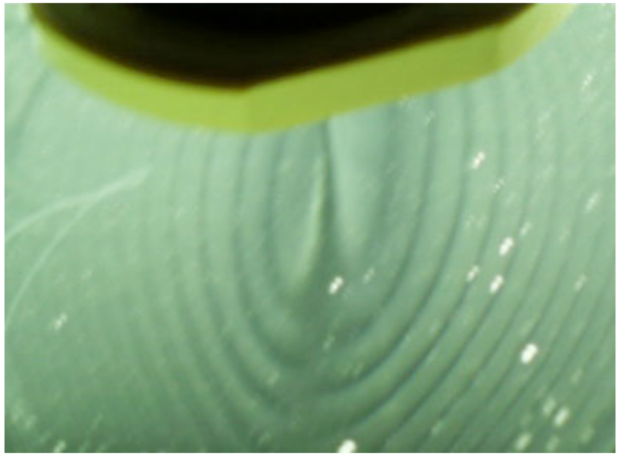
## *Limit 2: Low viscosity contrast* $(\eta_2/\eta_1 = 9.5)$

**edge-on view:**



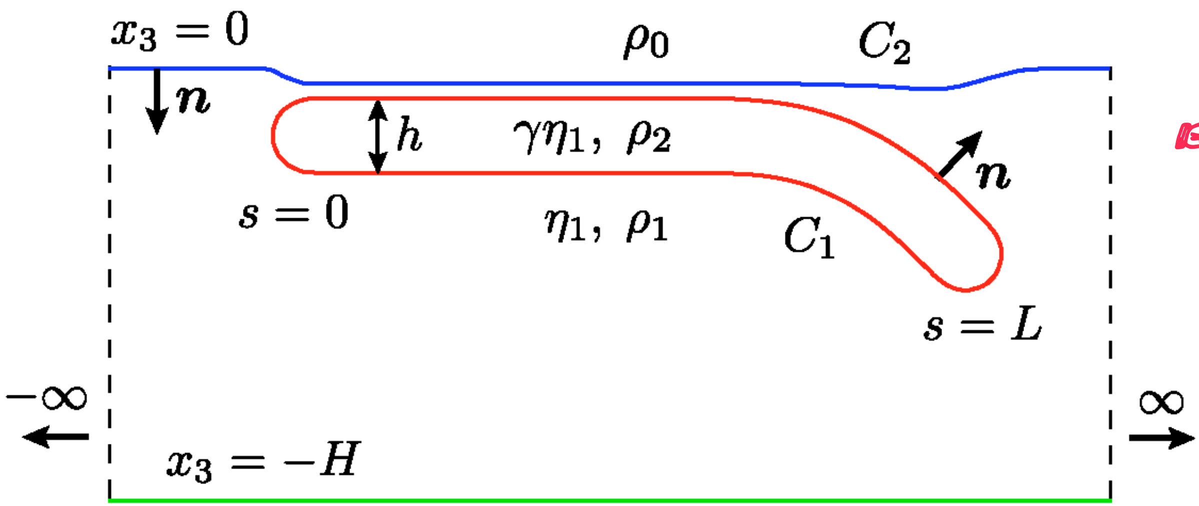

**side view:**


**from above:**



- ☞ folding suppressed
- ☞ small-amplitude waves propagate downward

## *Boundary-integral formulation for free subduction*




☞ see also poster of G. Morra et al.

➙ two coupled Fredholm integral equations for **u(x)** on the contours $C_1$ and $C_2$:

$$\frac{\rho_2 - \rho_1}{\eta_1} \int_{C_1} (\boldsymbol{g}\cdot\boldsymbol{y})\boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{J}(\boldsymbol{y}-\boldsymbol{x})\mathrm{d}\ell(\boldsymbol{y})$$

slab buoyancy

$$-\frac{\rho_1 - \rho_0}{\eta_1} \int_{C_2} (\boldsymbol{g}\cdot\boldsymbol{y})\boldsymbol{n}(\boldsymbol{y})\cdot\boldsymbol{J}(\boldsymbol{y}-\boldsymbol{x})\mathrm{d}\ell(\boldsymbol{y})$$

restoring force on topography

$$+(1-\gamma)\int_{C_1} \boldsymbol{u}(\boldsymbol{y})\cdot\boldsymbol{K}(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{y})\mathrm{d}\ell(\boldsymbol{y})$$

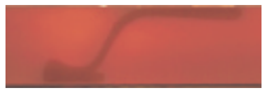
$$+\int_{C_2} \boldsymbol{u}_1(\boldsymbol{y})\cdot\boldsymbol{K}(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{y})\mathrm{d}\ell(\boldsymbol{y})$$

} double-layer integrals

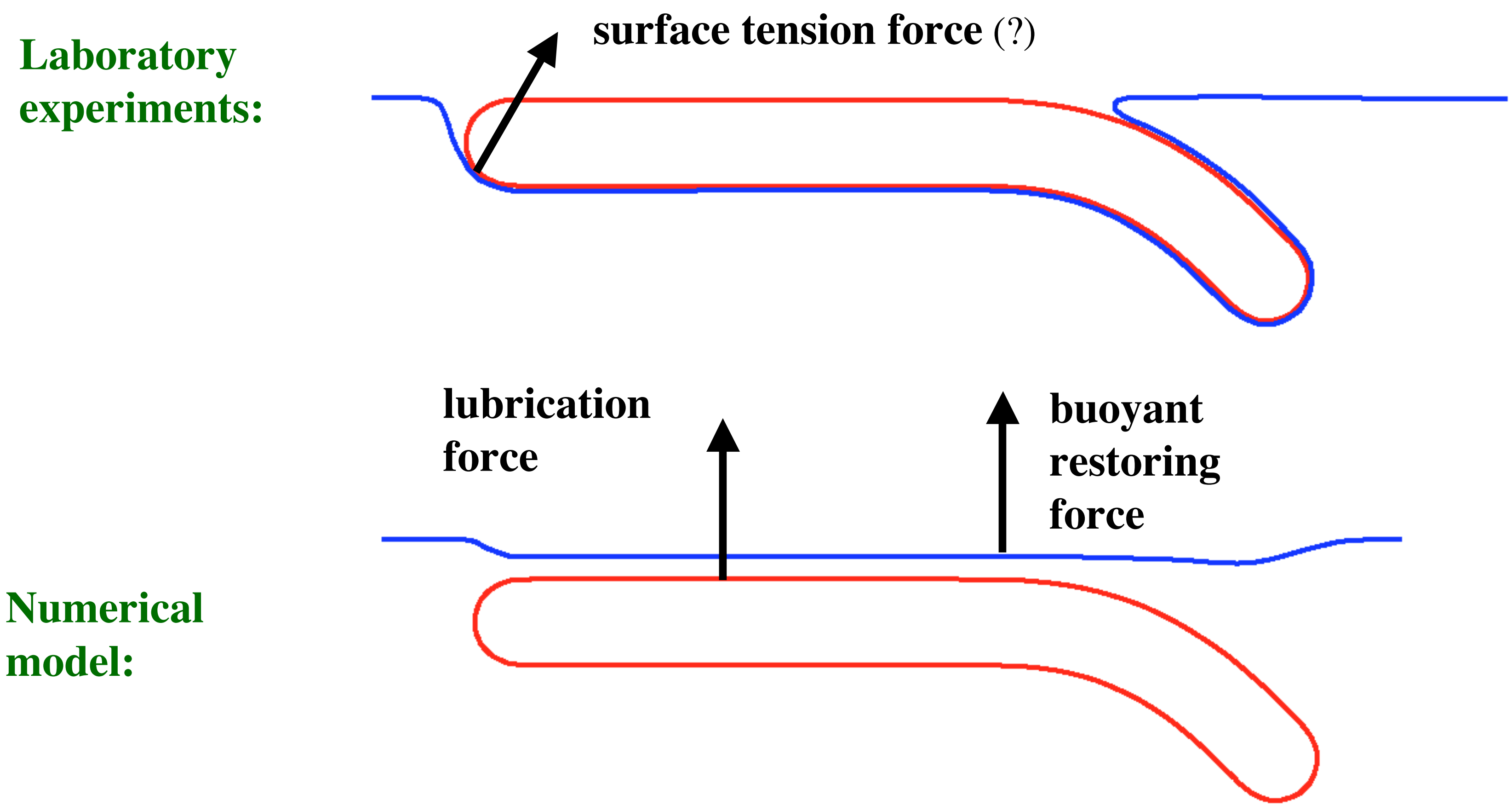
$$= \frac{1}{2}\begin{cases} (1+\gamma)\boldsymbol{u}(\boldsymbol{x}) & \boldsymbol{x} \in C_1 \\ \boldsymbol{u}_1(\boldsymbol{x}) & \boldsymbol{x} \in C_2 \end{cases}$$

### *Advantages:*

- Reduction of dimensionality (3D ➙ 2D or 2D ➙ 1D)
- True free surface
- No wall effects (unless desired)
- Accurate interface tracking
- Closely matches typical experimental configurations

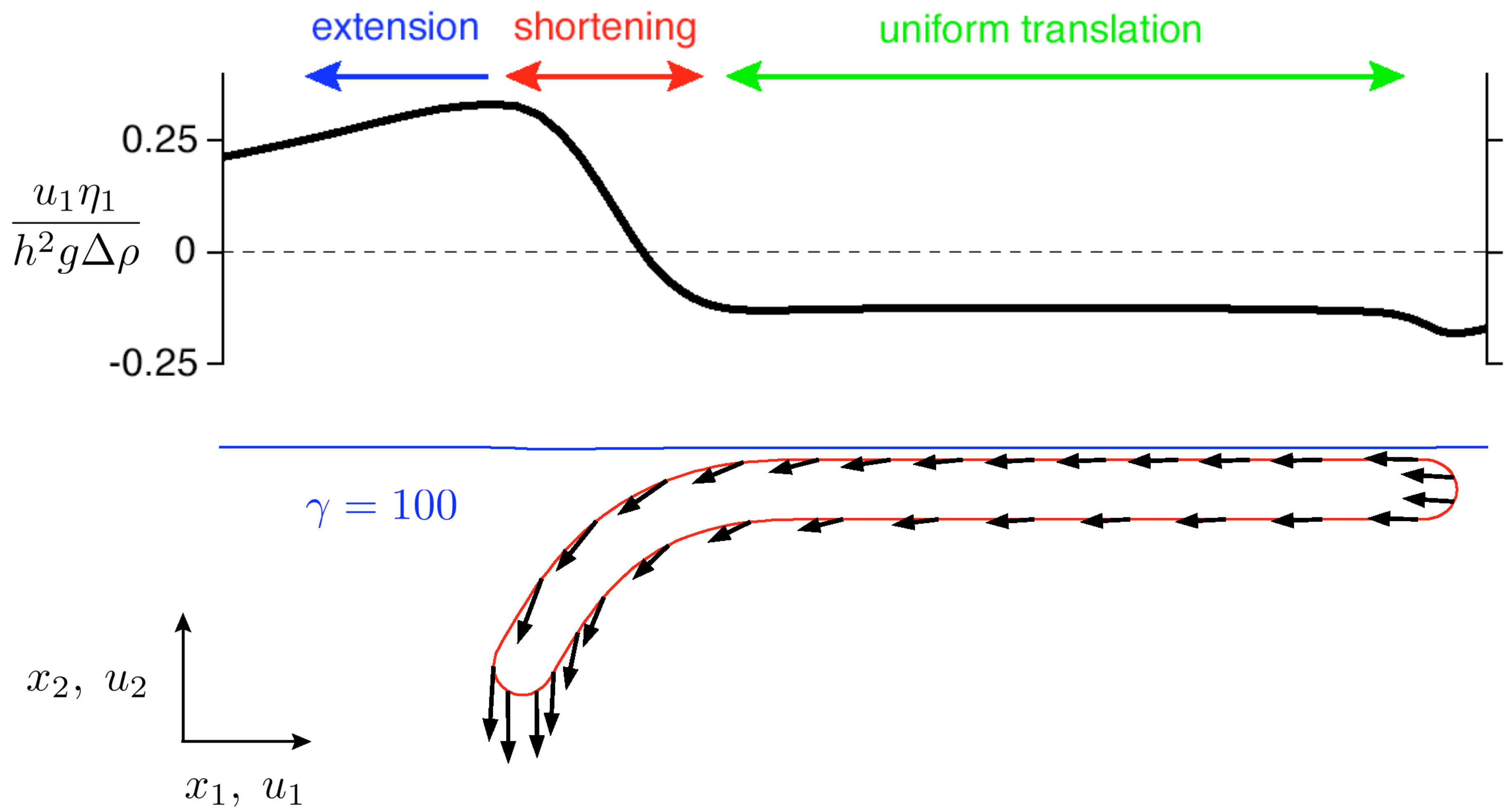
(e.g. Roma-III) 

*What holds the plate up?*

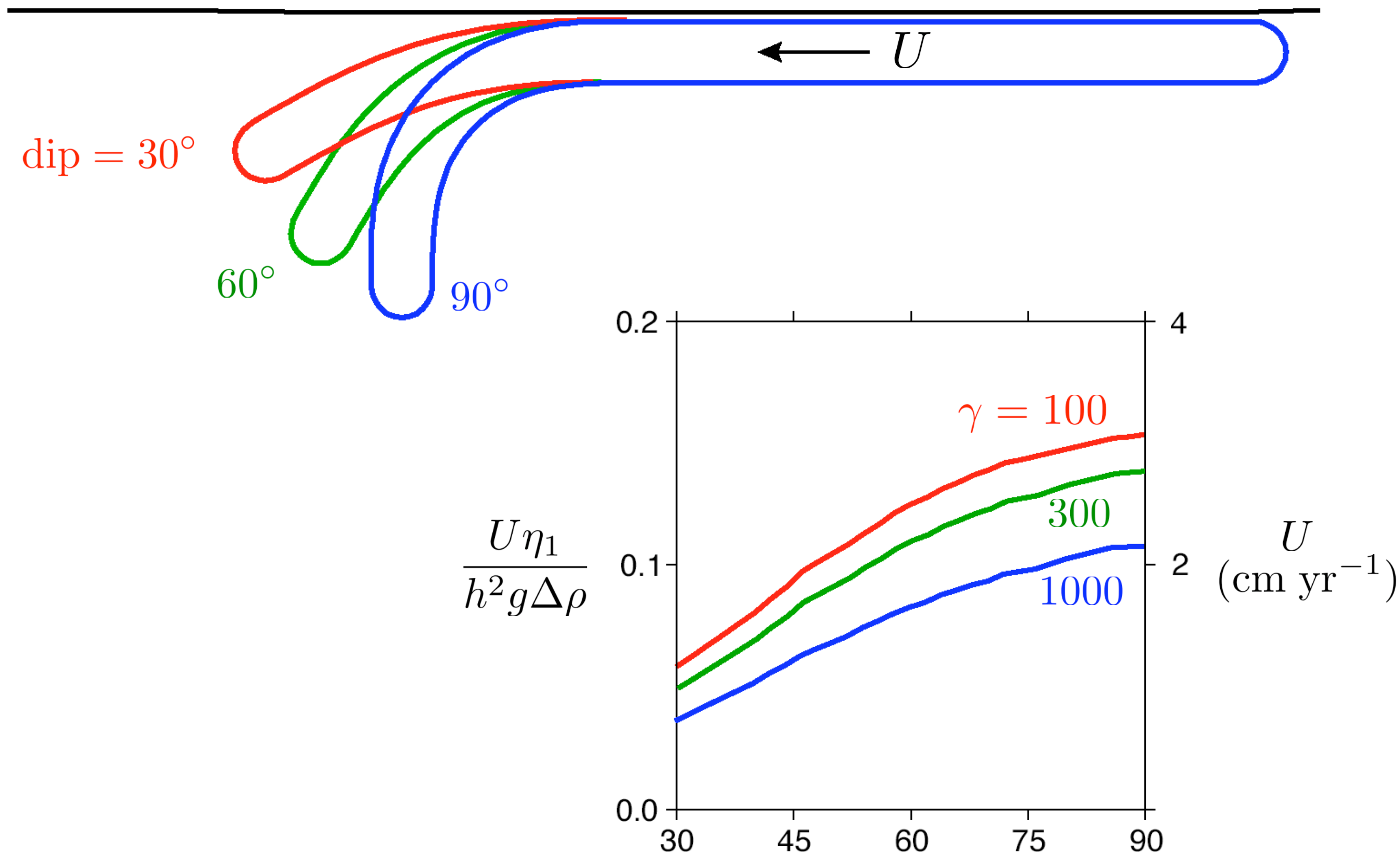


# *Example: 2D sheet subducting in a fluid half-space*

**Horizontal velocity of the free surface:**



## *Systematics :*
## *plate speed vs. slab dip and viscosity contrast*



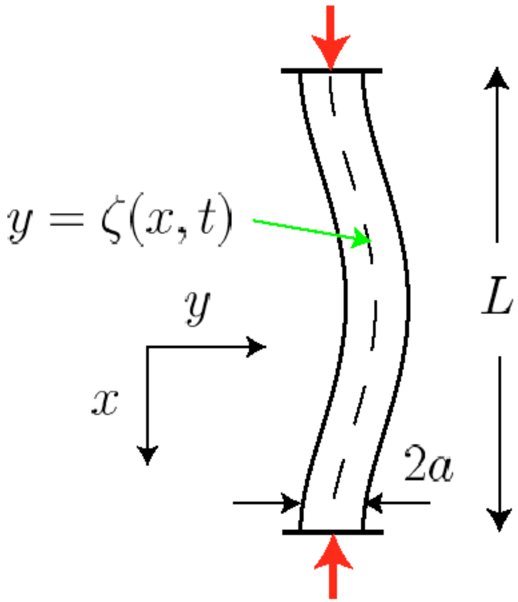
## *Work in Progress*

- Extension of the numerical approach to :
  - 3D flow
  - Mantle viscosity stratification

- Predictive scaling laws for key subduction parameters :
  - Plate speed
  - Trench rollback speed
  - Point of transition to back-arc extension
  - State of stress in the slab
  - Slab morphology (incl. buckling instabilities)

- Comparison with laboratory experiments (Roma-III)

# *Physical criterion for onset of buckling*




*Principle:* (a) compression is required to amplify shape perturbations (Taylor 1968)

(b) growth rate of perturbations must exceed thickening rate

*Experimental observation:*

Onset of buckling depends only on geometry, independently of viscosity ν and thickening rate Δ

*Mathematical analysis:*

(1) Torque balance in the absence of body forces: $$\frac{\partial}{\partial t}\frac{\partial^4 \zeta}{\partial x^4} = -\frac{4\Delta}{a^2}\frac{\partial^2 \zeta}{\partial x^2}$$

(2) Eigensolution for clamped ends: $$\zeta = A\left(1 + \cos\frac{2\pi x}{L}\right)\exp(\lambda t), \qquad \lambda = \left(\frac{L}{\pi a}\right)^2 \Delta$$

(3) Growth rate λ exceeds thickening rate Δ if $$\boxed{\frac{L}{a} > \pi}$$